# Optimal Parallel Routing in Star Networks

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**Abstract**

Star networks have been recently proposed as attractive alternatives to the popular hypercube for interconnecting processors on a parallel computer. In this paper, we present an efficient algorithm that constructs an optimal parallel routing in star networks. Our result improves previous results for the problem.

## 1 Introduction

The star network [2] has received considerable attention recently by researchers as a graph model for interconnection network. It has been shown that it is an attractive alternative to the widely used hypercube model. Like the hypercube, the star network is vertex- and edge-symmetric, strongly hierarchical, and maximally fault tolerant. Moreover, it has a smaller diameter and degree while comparing with hypercube of comparable number of vertices.

The rich structural properties of the star networks have been studied by many researchers. The $n$-star network $S_n$ is a degree $n-1$, $(n-1)$-connected, and vertex symmetric Cayley graph [1, 2]. Jwo, Lakshmivarahan, and Dhall [15] showed that the star networks are hamiltonian. Qiu, Akl, and Meijer [22] showed that the $n$-star network can be decomposed into $(n-1)!$ node-disjoint paths of length $n-1$, and can be decomposed into $(n-2)!$ node-disjoint cycles of length $(n-1)n$. Results in embedding hypercubes into star networks have been obtained by M. Nigam, S. Sahni, and B. Krishnamurthy [20] and by Miller, Pritikin, and Sudborough [18]. Broadcasting on star networks has also been studied recently [3, 4, 14, 23].

Routing on star networks was first studied by Akers and Krishnamurthy [2] who derived a formula for the length of the shortest path between any two nodes in a star network and developed an efficient algorithm for constructing such a path. Recently, parallel routing, i.e., constructing node-disjoint paths, on star networks has received much attention. Sur and Srimni [24] demonstrated that $n-1$ node-disjoint paths can be constructed between any two nodes in $S_n$ in polynomial time. Dietzfelbinger, Madhavapeddy, and Sudborough [11] derived an improved algorithm that constructs $n-1$ node-disjoint paths of length bounded by 4 plus the diameter of $S_n$. The algorithm was further improved by Day and Tripathi [10] who developed an efficient algorithm that constructs $n-1$ node-disjoint paths of length bounded by 4 plus the distance from $u$ to $v$ in $S_n$. The problem was also investigated by Jwo, Lakshmivarahan, and Dhall [16]. Misic and Jovanovic [19] derived a general algebraic expression for all (not necessarily node-disjoint) shortest paths between any two nodes in $S_n$. Palis and Rajasekaran [21], Dietzfelbinger, Madhavapeddy, and Sudborough [11], Qiu, Akl, and Meijer [22], and Chen and Chen [9] have considered the problem of node-disjoint paths between two sets of nodes in a star network.

In this paper, we will improve the previous results on node-to-node parallel routing in star networks by developing an efficient algorithm that constructs optimal parallel routing between any two nodes in a star network. More specifically, let $u$ and $v$ be any two nodes in the $n$-star network

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$S_n$ and let $dist(u, v)$ be the distance from $u$ to $v$ in $S_n$. The *bulk length* of a group of $n-1$ node-disjoint paths connecting $u$ and $v$ in $S_n$ is defined to be the length of the longest path in the group. Define the *bulk distance* $Bdist(u, v)$ between $u$ and $v$ to be the minimum bulk length over all groups of $n-1$ node-disjoint paths connecting $u$ and $v$ in $S_n$. We develop an $O(n^2 \log n)$ time algorithm that, given two nodes $u$ and $v$ in $S_n$, constructs a group of $n-1$ node-disjoint paths of bulk length $Bdist(u, v)$ that connect the nodes $u$ and $v$ in $S_n$.

Our algorithm involves careful analysis on the lower bound on the bulk distance $Bdist(u, v)$ between two nodes $u$ and $v$ in $S_n$, a non-trivial reduction from the parallel routing problem on star networks to a combinatorial problem called *Partition Matching*, a subtle solution to the Partition Matching problem, and a number of routing algorithms on different kinds of pairs of nodes in a star network. The basic idea of the algorithm can be roughly described as follows. Let $u$ and $v$ be two nodes in the $n$-star network $S_n$. According to Day and Tripathi [10], the bulk distance $Bdist(u, v)$ is equal to $dist(u, v) + b$, where $b = 0$, 2, or 4. We first derive a necessary and sufficient condition for a pair of nodes $u$ and $v$ to have bulk distance $dist(u, v) + 4$. For a pair $u$ and $v$ whose bulk distance is less than $dist(u, v) + 4$, we develop an efficient algorithm that constructs a group of $n-1$ node-disjoint paths of bulk length $dist(u, v) + 2$ between $u$ and $v$. Finally, an efficient algorithm, which is obtained from a reduction of an efficient algorithm solving the Partition Matching problem, is developed that constructs the maximum number of node-disjoint shortest paths (of length $dist(u, v)$) between $u$ and $v$. Combining all these analysis and algorithms gives us an efficient algorithm that constructs an optimal parallel routing on star networks. We should also point out that the running time of our algorithm is almost optimal (differs at most by a $\log n$ factor) since a lower bound $\Omega(n^2)$ on running time of parallel routing algorithms on star networks can be easily derived.

## 2 Preliminary

A permutation $u = a_1 a_2 \cdots a_n$ of the symbols $1, 2, \ldots, n$ can be given by a product of disjoint cycles [5], which is called the *cycle structure* of the permutation. A cycle is *nontrivial* if it contains more than one symbol. Otherwise the cycle is *trivial*. The cycle containing the symbol 1 will be called the *primary cycle* in the cycle structure. A $\pi[1, i]$ *transposition* on the permutation $u$ is to exchange the positions of the first symbol and the $i$th symbol in $u$: $\pi[1, i](u) = a_i a_2 a_3 \cdots a_{i-1} a_1 a_{i+1} \cdots a_n$. It is sometimes more convenient to write the transposition $\pi[1, i](u)$ on $u$ as $\pi[a_i](u)$ to indicate that the transposition exchanges the positions of the first symbol and the symbol $a_i$ in $u$. Let us consider how a transposition changes the cycle structure of a permutation. Write $u$ in its cycle structure

$$u = (a_{11} \cdots a_{1n_1}\ 1)(a_{21} \cdots a_{2n_2}) \cdots (a_{k1} \cdots a_{kn_k})$$

If $a_i$ is not in the primary cycle, then $\pi[1, i]$ "merges" the cycle containing $a_i$ into the primary cycle. More precisely, suppose that $a_i = a_{21}$ (note that each cycle can be cyclically permuted and the order of the cycles is irrelevant), then the permutation $\pi[1, i](u)$ will have the following cycle structure:

$$\pi[1, i](u) = (a_{21} \cdots a_{2n_2} a_{11} \cdots a_{1n_1}\ 1)(a_{31} \cdots a_{3n_3}) \cdots (a_{k1} \cdots a_{kn_k})$$

If $a_i$ is in the primary cycle, then $\pi[1, i]$ "splits" the primary cycle into two cycles. More precisely, suppose that $a_i = a_{1j}$, where $2 \leq j \leq n_1 + 1$ and we have let $a_{1n_1+1} = 1$ (note that $a_{11} = a_1$ and we assume $i > 1$), then $\pi[1, i](u)$ will have the following cycle structure:

$$\pi[1, i](u) = (a_{11} \cdots a_{1j-1})(a_{1j} \cdots a_{1n_1} 1)(a_{21} \cdots a_{2n_2}) \cdots (a_{k1} \cdots a_{kn_k})$$

In particular, if $a_i = a_{12}$, then we say that the transposition $\pi[1, i]$ "deletes" the symbol $a_{11}$ from the primary cycle.

Note that if a symbol $a_i$ is in a trivial cycle in the cycle structure of the permutation $u = a_1 a_1 \cdots a_n$, then the symbol is in its "correct" position, i.e., $a_i = i$, and that if a symbol is in a nontrivial cycle, then the symbol is not in its correct position. Denote by $\varepsilon$ the identity permutation $\varepsilon = 12 \cdots n$, in which every symbol is in a trivial cycle.

The *n-dimensional star network* (or simply the *n-star network*) $S_n$ is an undirected graph consisting of $n!$ nodes labeled with the $n!$ permutations on symbols $1, 2, \ldots, n$. There is an edge between two nodes $u$ and $v$ in $S_n$ if and only if there is a transposition $\pi[1, i]$, $2 \leq i \leq n$, such that $\pi[1, i](u) = v$. A path from a node $u$ to a node $v$ in the $n$-star network $S_n$ corresponds to a sequence of applications of the transpositions $\pi[1, i]$, $2 \leq i \leq n$, starting from the permutation $u$ and ending at the permutation $v$. A (node-to-node) *parallel routing* in $S_n$ is to construct the maximum number of node-disjoint paths between two given nodes in $S_n$. Since the $n$-star network $S_n$ is $(n-1)$-connected [2] and every node in $S_n$ is of degree $n-1$, by Menger's theorem [17], a parallel routing in $S_n$ should produce exactly $n-1$ node-disjoint paths between two given nodes. Moreover, since the $n$-star network is vertex symmetric [2], a parallel routing in the $n$-star network $S_n$ between any two given nodes can be easily mapped to a parallel routing between a node and the identity node $\varepsilon$ in $S_n$. We will write $dist(u)$ and $Bdist(u)$ for the distance $dist(u, \varepsilon)$ and the bulk distance $Bdist(u, \varepsilon)$, resepctively. By the definitions, we always have $Bdist(u) \geq dist(u)$.

Let $u$ be a node in the $n$-star network with cycle structure $u = c_1 \cdots c_k e_1 \cdots e_m$, where $c_i$ are nontrivial cycles and $e_j$ are trivial cycles. If we further let $l = \sum_{i=1}^{k} |c_i|$, where $|c_i|$ denotes the number of symbols in the cycle $c_i$, then the distance from the node $u$ to the identity node $\varepsilon$ is given by the following formula [2].

$$dist(u) = \begin{cases} l + k & \text{if the primary cycle is a trivial cycle} \\ l + k - 2 & \text{if the primary cycle is a nontrivial cycle} \end{cases}$$

Combining this formula with the above discussion on the effect of applying a transposition on a permutation, we derive the following necessary rules for tracing a shortest path from the node $u$ to the identity node $\varepsilon$ in the $n$-star network $S_n$.

**Shortest Path Rules**

*Rule 1.* If the primary cycle is a trivial cycle in $u$, then in the next node on any shortest path from $u$ to $\varepsilon$, a nontrivial cycle $c_i$ is merged into the primary cycle. This corresponds to applying transposition $\pi[a]$ to $u$ with $a \in c_i$;

*Rule 2.* If the primary cycle $c_1 = (a_{11} a_{12} \cdots a_{1n_1} a_{1n_1+1})$ is a nontrivial cycle in $u$, where $a_{1n_1+1} = 1$, then in the next node on any shortest path from $u$ to $\varepsilon$, either a nontrivial cycle $c_i \neq c_1$ is merged into the primary cycle (this corresponds to applying transposition $\pi[a]$ to $u$, where $a \in c_i$), or the symbol $a_{11}$ is deleted from the primary cycle $c_1$ (this corresponds to applying transposition $\pi[a_{12}]$ to $u$).

**Fact 2.1.** A shortest path from $u$ to $\varepsilon$ in $S_n$ is obtained by a sequence of applications of the Shortest Path Rules, starting from the permutation $u$.

**Fact 2.2.** If a symbol $a \neq 1$ is in a trivial cycle in $u$, then $a$ will stay in a trivial cycle in any node on a shortest path from $u$ to $\varepsilon$.

**Fact 2.3.** If an edge $[u, v]$ in $S_n$ does not lead to a shortest path from $u$ to $\varepsilon$, then $dist(v) = dist(u) + 1$. Consequently, let $P$ be a path from $u$ to $\varepsilon$ in which exactly $k$ edges do not follow the Shortest Path Rules, then the length of the path $P$ is equal to $dist(u) + 2k$.

**Fact 2.4.** [10] There are no cycles of odd length in an $n$-star network. Consequently, the length of any path from a node $u$ to the node $\varepsilon$ is equal to $dist(u)$ plus an even number.

Given a node $u = c_1 \ldots c_k e_1 \ldots e_m$ in the $n$-star network $S_n$, where $c_i$ are nontrivial cycles and $e_j$ are trivial cycles, a shortest path from $u$ to $\varepsilon$ can be constructed through the following two-stage process:

1. Merge in an arbitrary order each of the nontrivial cycles into the primary cycle. This will result in a node whose cycle structure has a single nontrivial cycle, which is the primary cycle. For example, suppose that $c_1 = (a_{11} \cdots a_{1n_1} 1)$ is the primary cycle in $u$ and that $c_i = (a_{i1} \cdots a_{in_i})$, $i = 2, \ldots, k$. Then by applying the transpositions $\pi[a_{21}]$, $\pi[a_{31}]$, …, $\pi[a_{k1}]$ in this order, we obtain a node with cycle structure
$$(a_{k1} \cdots a_{kn_k} a_{k-11} \cdots a_{k-1n_{k-1}} \cdots a_{21} \cdots a_{2n_2} a_{11} \cdots a_{1n_1} 1) e_1 \cdots e_m \tag{1}$$
2. Delete each of the symbols in the primary cycle until the node $\varepsilon$ is reached. For example, suppose that we start with the node with cycle structure as described in Equation (1), then we apply the transpositions $\pi[a_{k2}]$, …, $\pi[a_{kn_k}]$, $\pi[a_{k-11}]$, …, $\pi[a_{k-1n_{k-1}}]$, $\cdots$, $\pi[a_{21}]$, …, $\pi[a_{2n_2}]$, $\pi[a_{11}]$, …, $\pi[a_{1n_1}]$, $\pi[1]$, in this order, to reach the node $\varepsilon$.

The above process will be called the *Merge-Delete* process. It is easy to verify, using the Shortest Path Rules, that the Merge-Delete process produces a shortest path from the node $u$ to the node $\varepsilon$ in the $n$-star network $S_n$. The most important property of the Merge-Delete process is that in each node (except the node $\varepsilon$) of the constructed shortest path, the primary cycle is of form $(\cdots a_{1n_1} 1)$, where $a_{1n_1}$ is fixed for all nodes on the path.

Parallel routing from a node $u$ to $\varepsilon$ in $S_n$ is particularly simple when the primary cycle in $u$ is a trivial cycle [10, 16]. For completeness, we describe the routing process for this case here. Let $u$ be such a node in $S_n$. For each symbol $i$, $2 \le i \le n$, construct a path $P_i$ as follows. The path $P_i$ starts with the edge $[u, \pi[i](u)]$ followed by a shortest path from the node $\pi[i](u)$ to $\varepsilon$, which is obtained by applying the Merge-Delete process starting from the node $\pi[i](u)$. It is easy to verify that if $i$ is in a nontrivial cycle in $u$, then the path $P_i$ has length $dist(u)$, and that if $i$ is in a trivial cycle in $u$ then the path $P_i$ has length $dist(u) + 2$. On the other hand, if any symbol $i$, $2 \le i \le n$, is in a trivial cycle in $u$, then any path from $u$ to $\varepsilon$ via $\pi[i](u)$ has length at least $dist(u) + 2$. Since the node $u$ has degree $n - 1$, any group of $n - 1$ node-disjoint paths from $u$ to $\varepsilon$ contains a path whose first edge is $[u, \pi[i](u)]$. This implies that the bulk distance $Bdist(u)$ is at least $dist(u) + 2$ if there is a symbol $i$ in a trivial cycle in $u$, for $2 \le i \le n$. Therefore, to show that the constructed paths $P_i$, $2 \le i \le n$, have bulk length $Bdist(u)$, we only need to prove that all these paths are node-disjoint. In fact, since the first edge on $P_i$ is via $\pi[i](u)$ and the subpath from $\pi[i](u)$ to $\varepsilon$ on $P_i$ follows the shortest path obtained by the Merge-Delete process, it is easy to verify that for each path $P_i$, there is a symbol $b_i \in \{2, \ldots, n\}$ uniquely associated with the path $P_i$ such that for each interior node on the path $P_i$, the primary cycle is of form $(\cdots b_i 1)$. Therefore, no two of the $n - 1$ paths $P_1$, …, $P_{n-1}$ can share a common interior node, and the paths $P_i$, $2 \le i \le n$, are node-disjoint. This gives a group of $n - 1$ node-disjoint paths of bulk length $Bdist(u)$ from $u$ to $\varepsilon$.

Therefore, throughout the rest of this paper, we discuss the parallel routing problem in star networks based on the following assumption:

**Assumption 2.1.**

The node $u$ in the $n$-star network $S_n$ has cycle structure $u = c_1 \cdots c_k e_1 \cdots e_m$, where $c_i$ are nontrivial cycles and $e_j$ are trivial cycles, and the primary cycle $c_1 = (a_1 a_2 \cdots a_d 1)$ is a nontrivial cycle.

## 3 Nodes with bulk distance $dist(u) + 4$

Day and Tripathi [10] have developed a routing algorithm that constructs $n-1$ node-disjoint paths of bulk length $dist(u)+4$ from a node $u$ to $\varepsilon$ in the $n$-star $S_n$. The basic idea of Day-Tripathi's algorithm can be described as follows. For each symbol $a \in \{2, \ldots, n\}$, construct a path $P_a$ in which each node has a cycle of form $(\cdots a1)$. This cycle distinguishes the path $P_a$ from the other constructed paths. Let $u = c_1 \cdots c_k e_1 \cdots e_m$, $c_1 = (a_1 a_2 \cdots a_d 1)$, be the node as described in Assumption 2.1. We describe the path $P_a$ in three different cases.

**Case 1.** The symbol $a$ is in a nontrivial cycle $c_i$, for $i > 1$. Without loss of generality, let $c_i = (a' \cdots a'' a)$. In this case, the first four nodes on the path $P_a$ are $u_1 = u$, $u_2 = \pi[a'](u_1)$, $u_3 = \pi[1](u_2)$, and $u_4 = \pi[a_1](u_3)$. The rest of the path $P_a$ is constructed by applying the Merge-Delete process, starting from the node $u_4$.

**Case 2.** The symbol $a$ is in a trivial cycle $e_j = (a)$. Then the first four nodes on the path $P_a$ are $u_1 = u$, $u_2 = \pi[a](u_1)$, $u_3 = \pi[1](u_2)$, and $u_4 = \pi[a_1](u_3)$, and the rest of the path $P_a$ is obtained by applying the Merge-Delete process starting from the node $u_4$.

**Case 3.** The symbol $a$ is in the cycle $c_1$. Let $a = a_j$, where $1 \leq j \leq d$. Then the first four nodes on the path $P_a$ are $u_1 = u$, $u_2 = \pi[a_{j+1}](u_1)$, $u_3 = \pi[1](u_2)$, and $u_4 = \pi[a_1](u_3)$ (in case $j = d$, simply let $u_4 = u$ and discard the nodes $u_1$, $u_2$, and $u_3$), and the rest of the path $P_a$ is obtained by applying the Merge-Delete process starting from the node $u_4$.

It is easy to verify that in each of the above three cases, the nodes $u_2$ and $u_3$ on the path $P_a$ are not contained in any other constructed path $P_{a'}$ for $a' \neq a$. It is also easy to check that the fourth node $u_4$ on the path $P_a$ contains a cycle of form $(\cdots a1)$. Since the Merge-Delete process keeps the cycle pattern $(\cdots a1)$ along the path $P_a$, we conclude that all these $n-1$ constructed paths $P_a$, $a \in \{2, \ldots, n\}$, are node-disjoint. Finally, by examining each constructed path $P_a$, we find out that at most two edges on $P_a$ do not follow the Shortest Path Rules. By Fact 2.3, the path $P_a$ has length at most $dist(u)+4$. This completes the description of Day-Tripathi's algorithm.

Combining Day-Tripathi's algorithm and Fact 2.4, we conclude that for any node $u$ in the $n$-star network $S_n$, the bulk distance $Bdist(u)$ is equal to $dist(u) + b_u$, where $b_u = 0, 2$, or $4$. In this section, we derive a necessary and sufficient condition for a node $u$ to have bulk distance $dist(u)+4$.

Let $P$ be a path in $S_n$ from $u$ to $\varepsilon$. We say that the path $P$ *leaves $u$ with symbol $a$* if the second node on $P$ is $\pi[a](u)$, and we say that the path $P$ *enters $\varepsilon$ with symbol $a'$* if the node next to the last node $\varepsilon$ on $P$ is $\pi[a'](\varepsilon)$, which has a single nontrivial cycle $(a'1)$.

**Lemma 3.1** *Let $u = c_1 \cdots c_k e_1 \cdots e_m$, $c_1 = (a_1 a_2 \cdots a_d 1)$, be the node in the $n$-star network $S_n$ as described in Assumption 2.1. Suppose that $m > \min\{2^{k-2}, 1 + \sum_{i=2}^{k} |c_i|\}$. Then any group of $n-1$ node-disjoint paths from $u$ to $\varepsilon$ has bulk length at least $dist(u)+4$.*

Proof. Assume that $P_1, \ldots, P_{n-1}$ are $n-1$ node-disjoint paths from $u$ to $\varepsilon$ of bulk length bounded by $dist(u)+2$. We show that in this case we must have $m \leq \min\{2^{k-2}, 1 + \sum_{i=2}^{k} |c_i|\}$. First note that for each symbol $b$ that is in a trivial cycle in $u$, one of the paths $P_1, \ldots, P_{n-1}$ must leave $u$ with $b$, and one of the paths $P_1, \ldots, P_{n-1}$ must enter $\varepsilon$ with $b$.

Suppose that a path $P_i$ leaves $u$ with a symbol $b$ in a trivial cycle in $u$. By the Shortest Path Rules, the node $\pi[b](u)$ does not lead to a shortest path from $u$ to $\varepsilon$. By Fact 2.3, $dist(\pi[b](u)) = dist(u)+1$. On the other hand, the length of $P_i$ is bounded by $dist(u)+2$. Thus, starting from the node $\pi[b](u)$, the path $P_i$ must strictly follow the Shortest Path Rules. In particular, no node on the path $P_i$ (including the node $\pi[b](u)$) can contain a cycle of form $(\cdots b'1)$, where $b' \neq a_d$ and $b' \notin \bigcup_{i=2}^{k} c_i$. This implies that the path $P_i$ must enter $\varepsilon$ with a symbol in the set $\{a_d\} \cup \bigcup_{i=2}^{k} c_i$.

Since there are exactly $m$ of the paths $P_1, \ldots, P_{n-1}$ leaving with symbols in trivial cycles in $u$, and since all these paths are node-disjoint and must enter $\varepsilon$ with symbols in $\{a_d\} \cup \cup_{i=2}^{k} c_i$, we conclude that there are at least $m$ different symbols in the set $\{a_d\} \cup \bigcup_{i=2}^{k} c_i$. That is, $m \leq 1 + \sum_{i=2}^{k} |c_i|$.

Now again consider the path $P_i$ leaving $u$ with a symbol $b$ in a trivial cycle in $u$. Let $u_i$ be the first node on $P_i$ in which the primary cycle is a trivial cycle. Note that $u_i \neq u$ and $u_i \neq \pi[b](u)$. Moreover, since the nontrivial cycles in the node $\pi[b](u)$ are $c_1'$, $c_2$, $\cdots$, $c_k$, where $c_1' = (ba_1 \cdots a_d 1)$, and the path $P_i$ strictly follows the Shortest Path Rules after the node $\pi[b](u)$, every nontrivial cycle in the node $u_i$ is a cycle in the set $\{c_2, \ldots, c_k\}$. Therefore, the node $u_i$, hence the path $P_i$, corresponds to a subset of the set $\{c_2, \ldots, c_k\}$.

Similarly, suppose that $P_h$ is a path among $P_1, \ldots, P_{n-1}$ that enters $\varepsilon$ with a symbol $b$ in a trivial cycle in $u$. Since at node $u$, the symbol $b$ is in a trivial cycle while at node $\pi[b](\varepsilon)$ on the path $P_h$, the symbol $b$ is in a nontrivial cycle, we can find two consecutive nodes $u_h$ and $v_h$ on the path $P_h$ such that the symbol $b$ is in a trivial cycle in the node $u_h$ while in a nontrivial cycle in the node $v_h$. By Fact 2.2, the edge $[u_h, v_h]$ on the path $P_h$ does not follow the Shortest Path Rules. By our assumption, the length of the path $P_h$ is bounded by $dist(u) + 2$. We conclude that the subpaths of $P_h$ from $u$ to $u_h$ and from $v_h$ to $\varepsilon$, respectively, must strictly follow the Shortest Path Rules. Note that by our assumption on the nodes $u_h$ and $v_h$, we must have $v_h = \pi[b](u_h)$, and the symbols $b$ and 1 are in the same cycle in the node $v_h$. Since the node $\pi[b](\varepsilon) = (b\ 1)$ (we discard trivial cycles) is on the subpath of $P_h$ from $v_h$ to $\varepsilon$, in order to let the subpath of $P_h$ from $v_h$ to $\varepsilon$ to strictly follow the Shortest Path Rules, the node $v_h$ must have a cycle of form $(b\ 1)$. Consequently, the symbol 1 is in a trivial cycle in the node $u_h$. Now since the subpath of $P_h$ from $u$ to $u_h$ follows strictly the Shortest Path Rules, every nontrivial cycle in the node $u_h$ must be one of the nontrivial cycles $c_2, \ldots, c_k$ in the node $u$. This proves that the node $u_h$, thus the path $P_h$, corresponds to a subset of the set $\{c_2, \ldots, c_k\}$.

As we have shown before, no path in $P_1, \ldots, P_{n-1}$ can both leave $u$ with a symbol $b$ and enters $\varepsilon$ with a symbol $b'$, where $b$ and $b'$ are symbols in trivial cycles in the node $u$. Therefore, there are exactly $2m$ paths among $P_1, \ldots, P_{n-1}$ that either leaves $u$ or enters $\varepsilon$ with a symbol in a trivial cycle in $u$. Each of these $2m$ paths corresponds to a subset in the set $\{c_2, \ldots, c_k\}$. Since all these paths are node-disjoint, we conclude that there are at least $2m$ different subsets of the set $\{c_2, \ldots, c_k\}$. That is, $2^{k-1} \geq 2m$, or equivalently, $m \leq 2^{k-2}$. □

Combined with Day-Tripathi's algorithm, Lemma 3.1 provides a sufficient condition for a node $u$ in the $n$-star network $S_n$ to have bulk distance $dist(u) + 4$. In the following, we show that this condition is also necessary. For this, we demonstrate that when $m \leq \min\{2^{k-2}, 1 + \sum_{i=2}^{k} |c_i|\}$, we can always construct $n-1$ node-disjoint paths from $u$ to $\varepsilon$ of bulk length $dist(u) + 2$.

We first make some conventions. We assume that $n \geq 3$ since the parallel routing on an $n$-star network $S_n$ for $n \leq 2$ is trivial. Let $c$ be a cycle, given in any fixed cyclic order. We will denote by $[c]$ the sequence of the symbols in the cycle $c$. Therefore, $([c]\ a_1 \cdots a_d)$ is a cycle starting with the symbols in the cycle $c$ followed by the symbols $a_1, \ldots, a_d$. Recall that a cycle can be arbitrarily cyclically rotated, still resulting in the same cycle. In many cases, we will discard irrelevant trivial cycles in a cycle structure.

Let $u = c_1 \cdots c_k e_1 \cdots e_m$, $c_1 = (a_1 a_2 \cdots a_d 1)$, be the node in the $n$-star network $S_n$ as described in Assumption 2.1, and suppose $m \leq \min\{2^{k-2}, 1 + \sum_{i=2}^{k} |c_i|\}$. We describe our construction in three different cases.

**Case I.** The number $k$ of nontrivial cycles in $u$ is equal to 1.

By the condition $m \leq 2^{k-2}$, we obtain $m = 0$. Therefore, the cycle structure of the node $u$ consists of a single (nontrivial) cycle $u = (a_1 \cdots a_{n-1} 1)$. Note that the unique cycle in $u$ contains at least 3 symbols since $n \geq 3$. The $n-1$ node-disjoint paths from $u$ to $\varepsilon$ are given as follows.

A path $P_1$ leaving $u$ with $a_2$ and entering $\varepsilon$ with $a_{n-1}$ is given by

$$(a_1 a_2 \cdots a_{n-1} 1) \to (a_2 \cdots a_{n-1} 1) \to \overset{D}{\cdots} \to (a_{n-1} 1) \to \varepsilon$$

where $\to \overset{D}{\cdots} \to$ stands for a sequence of transpositions that repeatedly deletes a symbol from the primary cycle.

A path $P_2$ leaving $u$ with 1 and entering $\varepsilon$ with $a_1$ is given by

$$(a_1 a_2 \cdots a_{n-1} 1) \to (a_1 a_2 \cdots a_{n-1})(1) \to (a_2 \cdots a_{n-1} a_1 1) \to \overset{D}{\cdots} \to (a_1 1) \to \varepsilon$$

For $j = 3, \ldots, n-1$, there is a path $P_j$ leaving $u$ with $a_j$ and entering $\varepsilon$ with $a_{j-1}$, given by

$$(a_1 \cdots a_{j-1} a_j \cdots a_{n-1} 1) \to (a_1 \cdots a_{j-1})(a_j \cdots a_{n-1} 1) \to \overset{D}{\cdots} \to (a_1 \cdots a_{j-1})(a_{n-1} 1) \to$$
$$\to (a_1 \cdots a_{j-1})(1) \to (a_1 \cdots a_{j-1} 1) \to \overset{D}{\cdots} \to (a_{j-1} 1) \to \varepsilon$$

It is easy to verify that each path $P_j$, $j = 1, \ldots, n-1$, has at most one edge that does not follow the Shortest Path Rules. Thus, all these constructed paths have length at most $dist(u) + 2$. The path $P_1$ keeps a cycle of form $(\cdots a_{n-1} 1)$. The path $P_2$ keeps a cycle of form $(\cdots a_1 1)$. For $j = 3, \ldots, n-1$, the first part of the path $P_j$ keeps a distinguished cycle $(a_1 \cdots a_{j-1})$ and the second part of the path $P_j$ keeps a cycle of form $(\cdots a_{j-1} 1)$. Therefore, all these constructed paths are node-disjoint. This gives in this case a group of $n-1$ node-disjoint paths of bulk length bounded by $dist(u) + 2$ from $u$ to $\varepsilon$.

**Case II.** The number $k$ of nontrivial cycles in $u$ is at least 2, and the number $m$ of trivial cycles in $u$ is 0.

In this case, the node $u$ can be written as $u = (a_1 \cdots a_d 1) c_2 \cdots c_k$.

A path $P_1$ leaving $u$ with 1 and entering $\varepsilon$ with $a_d$ is given by

$$(a_1 \cdots a_d 1) c_2 \cdots c_k \to (a_1 \cdots a_d)(1) c_2 \cdots c_k \to \overset{M}{\cdots} \to (a_1 \cdots a_d)([c_2] \cdots [c_k]\ 1) \to$$
$$\to \overset{D}{\cdots} \to (a_1 \cdots a_d)(1) \to (a_1 \cdots a_d 1) \to \overset{D}{\cdots} \to (a_d 1) \to \varepsilon$$

where $\to \overset{M}{\cdots} \to$ stands for a sequence of transpositions that repeatedly merges a nontrivial cycle into the primary cycle.

For $j = 2, \ldots, d$, there is a path $P_j$ leaving $u$ with $a_j$ and entering $\varepsilon$ with $a_{j-1}$, given by

$$(a_1 \cdots a_{j-1} a_j \cdots a_d 1) c_2 \cdots c_k \to (a_1 \cdots a_{j-1})(a_j \cdots a_d 1) c_2 \cdots c_k \to \overset{D}{\cdots} \to$$
$$\to (a_1 \cdots a_{j-1})(1) c_2 \cdots c_k \to (a_1 \cdots a_{j-1} 1) c_2 \cdots c_k \to \overset{M}{\cdots} \to$$
$$\to ([c_2] \cdots [c_k]\ a_1 \cdots a_{j-1} 1) \to \overset{D}{\cdots} \to (a_{j-1} 1) \to \varepsilon$$

Note that this group of paths is constructed only when $d \geq 2$.

For each symbol $a \in \cup_{i=2}^{k} c_i$, there is a path $P_a$ leaving $u$ with $a$ and entering $\varepsilon$ with $a$. Since each nontrivial cycle can be cyclically rotated and the order of the cycles $c_2$, …, $c_k$ is irrelevant, we can assume, without loss of generality, that $c_2 = (aa' \cdots a'')$. The path $P_a$ is given by

$$(a_1 \cdots a_d 1)(aa' \cdots a'') c_3 \cdots c_k \to (aa' \cdots a'' a_1 \cdots a_d 1) c_3 \cdots c_k \to$$
$$\to (aa' \cdots a'' a_1 \cdots a_d)(1) c_3 \cdots c_k \to (a' \cdots a'' a_1 \cdots a_d a 1) c_3 \cdots c_k \to$$
$$\to \overset{M}{\cdots} \to ([c_3] \cdots [c_k]\ a' \cdots a'' a_1 \cdots a_d a 1) \to \overset{D}{\cdots} \to (a 1) \to \varepsilon$$

It is again easy to verify that all these $n-1$ paths are of length at most $dist(u)+2$ and node-disjoint. Thus, this gives in this case a group of $n-1$ node-disjoint paths of bulk length bounded by $dist(u)+2$ from $u$ to $\varepsilon$.

**Case III.** The number $k$ of nontrivial cycles in $u$ is at least 2 and the number $m$ of trivial cycles in $u$ is at least 1.

In this case, the node $u$ can be written as $u = (a_1 \cdots a_d 1)c_2 \cdots c_3(b_1)\cdots(b_m)$, where $(a_1 \cdots a_d 1)$, $c_2$, …, $c_k$ are nontrivial cycles and $(b_1)$, …, $(b_m)$ are trivial cycles.

A path $P_1$ leaving $u$ with $b_1$ and entering $\varepsilon$ with $a_d$ is constructed as follows.

$$\begin{aligned}(a_1 \cdots a_d 1)c_2 \cdots c_k(b_1)\cdots(b_m) \to (b_1 a_1 \cdots a_d 1)c_2 \cdots c_k(b_2)\cdots(b_m) \to \overset{M}{\cdots} \to \\ \to ([c_2]\cdots[c_k]\; b_1 a_1 \cdots a_d 1) \to \overset{D}{\cdots} \to (a_d 1) \to \varepsilon\end{aligned}$$

A path $P_1'$ leaving $u$ with $a_2$ and entering $\varepsilon$ with $b_1$ is given by

$$\begin{aligned}(a_1 a_2 \cdots a_d 1)c_2 \cdots c_k(b_1)\cdots(b_m) \to (a_2 \cdots a_d 1)c_2 \cdots c_k(b_1)\cdots(b_m) \to \overset{D}{\cdots} \to \\ \to (1)c_2 \cdots c_k(b_1)\cdots(b_m) \to (b_1 1)c_2 \cdots c_k \to \overset{M}{\cdots} \to ([c_2]\cdots[c_k]\; b_1 1) \to \overset{D}{\cdots} \to (b_1 1) \to \varepsilon\end{aligned}$$

If $d=1$, then the above constructed path $P_1'$ leaves $u$ with the symbol 1 and enters $\varepsilon$ with $b_1$.

For each $i$, $i = 2, \ldots, m$, we construct two paths $P_i$ and $P_i'$ as follows. First mark all symbols in $\cup_{i=2}^{k} c_i$ as "unused", and mark all subsets of the set $S = \{c_2, \ldots, c_k\}$ as "unused". Then for each $i = 2, \ldots, m$, pick an unused symbol $a_i'$ in $\cup_{i=2}^{k} c_i$, and an unused subset $S_i = \{c_2^{(i)}, \ldots, c_{h_i}^{(i)}\}$ from the set $S = \{c_2, \ldots, c_k\}$ such that $S_i \neq S$ and $a_i'$ is contained in a cycle in the subset $S_i$. Let $\{c_{h_i+1}^{(i)}, \ldots, c_k^{(i)}\} = S - S_i$. Without loss of generality, we assume that the cycle $c_{h_i}^{(i)}$ contains the symbol $a_i'$: $c_{h_i}^{(i)} = (a_i' a' \cdots a'')$. The path $P_i$ leaves $u$ with $a_i'$ and enters $\varepsilon$ with $b_i$, and is given by

$$\begin{aligned}(a_1 \cdots a_d 1)c_2 \cdots c_k(b_1)\cdots(b_m) = (a_1 \cdots a_d 1)c_2^{(i)} \cdots c_{h_i}^{(i)} c_{h_i+1}^{(i)} \cdots c_k^{(i)}(b_1)\cdots(b_m) \to \\ \to (a_i' a' \cdots a'' a_1 \cdots a_d 1)c_2^{(i)} \cdots c_{h_i-1}^{(i)} c_{h_i+1}^{(i)} \cdots c_k^{(i)}(b_1)\cdots(b_m) \to \overset{M}{\cdots} \to \\ \to ([c_2^{(i)}]\cdots[c_{h_i-1}^{(i)}] a_i' a' \cdots a'' a_1 \cdots a_d 1)c_{h_i+1}^{(i)} \cdots c_k^{(i)}(b_1)\cdots(b_m) \to \overset{D}{\cdots} \to \\ \to (a_d 1)c_{h_i+1}^{(i)} \cdots c_k^{(i)}(b_1)\cdots(b_m) \to (1)c_{h_i+1}^{(i)} \cdots c_k^{(i)}(b_1)\cdots(b_m) \to (b_i 1)c_{h_i+1}^{(i)} \cdots c_k^{(i)} \to \\ \to \overset{M}{\cdots} \to ([c_{h_i+1}^{(i)}]\cdots[c_k^{(i)}]\; b_i 1) \to \overset{D}{\cdots} \to (b_i 1) \to \varepsilon\end{aligned}$$

and the path $P_i'$ leaves $u$ with $b_i$ and enters $\varepsilon$ with $a_i'$, and is given by

$$\begin{aligned}(a_1 \cdots a_d 1)c_2 \cdots c_k(b_1)\cdots(b_m) = (a_1 \cdots a_d 1)c_2^{(i)} \cdots c_{h_i}^{(i)} c_{h_i+1}^{(i)} \cdots c_k^{(i)}(b_1)\cdots(b_m) \to \\ \to (b_i a_1 \cdots a_d 1)c_2^{(i)} \cdots c_{h_i}^{(i)} c_{h_i+1}^{(i)} \cdots c_k^{(i)} \to \overset{M}{\cdots} \to ([c_{h_i+1}^{(i)}]\cdots[c_k^{(i)}]\; b_i a_1 \cdots a_d 1)c_2^{(i)} \cdots c_{h_i}^{(i)} \to \\ \to \overset{D}{\cdots} \to (1)c_2^{(i)} \cdots c_{h_i}^{(i)} \to (a' \cdots a'' a_i' 1)c_2^{(i)} \cdots c_{h_i-1}^{(i)} \to \overset{M}{\cdots} \to \\ \to ([c_2^{(i)}]\cdots[c_{h_i-1}^{(i)}]\; a' \cdots a'' a_i' 1) \to \overset{D}{\cdots} \to (a_i' 1) \to \varepsilon\end{aligned}$$

Now mark the symbol $a_i'$ and the subsets $\{c_2^{(i)}, \ldots, c_{h_i}^{(i)}\}$ and $\{c_{h_i+1}^{(i)}, \ldots, c_k^{(i)}\}$ of the set $\{c_2, \ldots, c_k\}$ as "used".

We must justify that the construction of the paths $P_i$ and $P_i'$ is always possible. Since $m \leq 1 + \sum_{i=2}^{k} |c_i|$, there are at least $m-1$ symbols in $\cup_{i=2}^{k} c_i$. Therefore, for each $i$, $i = 2, \ldots, m$, we should always be able to find an unused symbol $a_i'$ in $\cup_{i=2}^k c_i$. Now fix the symbol $a_i'$. There are $2^{k-2}-1$ different subsets $S'$ of the set $S = \{c_2, \ldots, c_k\}$ such that $S' \neq S$ and a cycle in $S'$ contains the symbol $a_i'$. Since $m \leq 2^{k-2}$, at least one of such subsets has not been used for constructing the previous paths $P_j$ and $P_j'$, $j = 2, \ldots, i-1$ (note that in the two subsets $\{c_2^{(j)}, \ldots, c_{h_j}^{(j)}\}$ and $\{c_{h_j+1}^{(j)}, \ldots, c_k^{(j)}\}$ used for the paths $P_j$ and $P_j'$, only one of them has a cycle containing the symbol $a_i'$). Also note that if $S'$ is an unused subset of $S$, then $S - S'$ is also an unused subset of $S$. Therefore, we are always able to find an unused subset $S_i$ of $S$ such that $S_i \neq S$ and a cycle in $S_i$ contains the symbol $a_i'$. This ensures the possibility of the construction of the paths $P_i$ and $P_i'$, for all $i = 2, \ldots, m$.

If the cycle $c_1 = (a_1 \cdots a_d 1)$ contains more than two symbols, i.e., $d \geq 2$, then we also construct the following $d-1$ paths $Q_2$, …, $Q_d$.

The path $Q_2$ leaves $u$ with the symbol 1 and enters $\varepsilon$ with $a_1$:

$$\begin{aligned}(a_1a_2\cdots a_d1)c_2\cdots c_k &\to (a_1a_2\cdots a_d)(1)c_2\cdots c_k \to (a_2\cdots a_da_11)c_2\cdots c_k \to \\ &\to \overset{M}{\cdots} \to ([c_2]\cdots[c_k]\ a_2\cdots a_da_11) \to \overset{D}{\cdots} \to (a_11) \to \varepsilon\end{aligned}$$

and for $j = 3, \ldots, d$, the path $Q_j$ leaves $u$ with $a_j$ and enters $\varepsilon$ with $a_{j-1}$:

$$\begin{aligned}(a_1\cdots a_{j-1}a_j\cdots a_d1)c_2\cdots c_k &\to (a_1\cdots a_{j-1})(a_j\cdots a_d1)c_2\cdots c_k \to \overset{D}{\cdots} \to \\ &\to (a_1\cdots a_{j-1})(1)c_2\cdots c_k \to (a_1\cdots a_{j-1}1)c_2\cdots c_k \to \overset{M}{\cdots} \to \\ &\to ([c_2]\cdots[c_k]\ a_1\cdots a_{j-1}1) \to \overset{D}{\cdots} \to (a_{j-1}1) \to \varepsilon\end{aligned}$$

Finally, for each symbol $a$ in $\cup_{i=2}^k c_i$ that is not used in constructing the paths $P_i$ and $P_i'$, $i = 2, \ldots, m$, we construct a path $P_a$ leaving $u$ with $a$ and entering $\varepsilon$ with $a$. Without loss of generality, let the symbol $a$ be in the cycle $c_2$: $c_2 = (aa'\cdots a'')$.

$$\begin{aligned}(a_1\cdots a_d1)(aa'\cdots a'')c_3\cdots c_k &\to (aa'\cdots a''a_1\cdots a_d1)c_3\cdots c_k \to \\ \to (aa'\cdots a''a_1\cdots a_d)(1)c_3\cdots c_k &\to (a'\cdots a''a_1\cdots a_da1)c_3\cdots c_k \to \\ \to \overset{M}{\cdots} \to ([c_3]\cdots[c_k]\ a'\cdots a''a_1\cdots a_da1) &\to \overset{D}{\cdots} \to (a1) \to \varepsilon\end{aligned}$$

The above process constructs $n-1$ paths from the node $u$ to $\varepsilon$. It is easy to verify that each of the constructed paths has at most one edge that does not follow the Shortest Path Rules. By Fact 2.3, all these $n-1$ paths are of length at most $dist(u) + 2$. What remains is to show that all these paths are node-disjoint.

On each of the constructed paths, the nodes have a special cycle pattern to ensure that the path is node-disjoint from the other paths.

1. Path $P_1$ consists of two parts of different format: each node in the first part has a cycle of form $(\cdots b_1a_1\cdots a_d1)$, and each node in the second part has a single nontrivial cycle of form $(\cdots a_d1)$, where the symbol $a_d$ is uniquely associated with the path $P_1$;

2. Path $P_1'$ consists of two parts of different format: each node in the first part has a cycle structure of form $(\cdots 1)c_2\cdots c_k$, and each node in the second part has a cycle of form $(\cdots b_11)$, where the symbol $b_1$ is uniquely associated with the path $P_1'$;

3. Path $P_i$, $i = 2, \ldots, m$, consisits of three parts of different format: each node in the first part has a cycle of form $(\cdots a_i'a'\cdots a''a_1\cdots a_d1)$, each node in the second part has a cycle structure

of form $(\cdots 1)c^{(i)}_{h_i+1}\cdots c^{(i)}_k$, where the subset $\{c^{(i)}_{h_i+1},\ldots,c^{(i)}_k\}$ is uniquely associated with the path $P_i$, and each node in the third part has a cycle of form $(\cdots b_i 1)$, where the symbol $b_i$ is uniquely associated with the path $P_i$. Note that the assumption $S_i \neq S$ is necessary here — otherwise, the path $P_i$ would enter $\varepsilon$ with the symbol $a_d$ and would share a common interior node with path $P_1$;

Similarly, path $P'_i$, $i = 2, \ldots m$, consists of three parts of different format: each node in the first part has a cycle of form $(\cdots b_i a_1 \cdots a_d 1)$, each node in the second part has cycle structure of form $(\cdots 1)c^{(i)}_2\cdots c^{(i)}_{h_i}$, where the subset $\{c^{(i)}_2,\ldots,c^{(i)}_{h_i}\}$ is uniquely associated with the path $P'_i$, and each node in the third part has a cycle of form $(\cdots a'_i 1)$, where the symbol $a'_i$ is uniquely associated with the path $P'_i$. Again here the condition $S_i \neq S$ is necessary — otherwise, the third node of the path $P'_i$ would be the node $u$ again and the path $P'_i$ would then leave $u$ with the symbol $a_2$ and share an interior node with the path $P'_1$.

4. The second node of the path $Q_2$ has a distinguished cycle $(a_1 \cdots a_d)$, and the rest of the nodes in the path $Q_2$ have a cycle of form $(\cdots a_1 1)$, where the symbol $a_1$ is uniquely associated with the path $Q_2$;

5. The path $Q_j$, $j = 3, \ldots, d$, consists of two parts of different format: each node in the first part has a distinguished cycle $(a_1 \cdots a_{j-1})$, and each node in the second part has a cycle of form $(\cdots a_{j-1} 1)$, where the symbol $a_{j-1}$ is uniquely associated with the path $Q_j$;

6. For each symbol $a$ that is not used in the paths $P_i$ and $P'_i$, $i = 2, \ldots, m$, the second node in the path $P_a$ has a cycle $(aa' \cdots a'' a_1 \cdots a_d 1)$, the third node in the path $P_a$ has a cycle $(aa' \cdots a'' a_1 \cdots a_d)$, and the rest of the nodes in $P_a$ have a cycle of form $(\cdots a 1)$, where the symbol $a$ is uniquely associated with the path $P_a$.

Therefore, all these $n-1$ constructed paths are node-disjoint. Summarizing all the above discussions, we obtain the following lemma.

**Lemma 3.2** *Let $u = c_1 \cdots c_k e_1 \cdots e_m$, $c_1 = (a_1 a_2 \cdots a_d 1)$, be the node in the $n$-star network $S_n$ as described in Assumption 2.1. If $m \leq \min\{2^{k-2}, 1 + \sum_{i=2}^{k} |c_i|\}$, then a group of $n-1$ node-disjoint paths of bulk length $dist(u) + 2$ from $u$ to $\varepsilon$ can be constructed in time $O(n^2 \log n)$.*

Proof. The construction of a group of $n-1$ node-disjoint paths of bulk length $dist(u)+2$ from $u$ to $\varepsilon$ has been given in the above discussion.

For each $P$ of the paths $P_1$, $P'_1$, $Q_j$, $j = 2, \ldots, d$, and $P_a$, where $a$ is a symbol in $\cup_{i=2}^{k} c_i$ that is not used in the construction of the paths $P_i$ and $P'_i$, $i = 2, \ldots, m$, the construction of $P$ can be obviously implemented in time $O(n)$. (Note that according to [2], $dist(u) = O(n)$.)

To construct the paths $P_i$ and $P'_i$, we need to find an unused symbol $a'_i$. The symbol $a'_i$ can be simply found in time $O(n)$ if we keep a record for each symbol to indicate whether it has been used. Once the symbol $a'_i$ is decided, we need to find an unused subset $S_i$ of the set $S = \{c_2, \ldots, c_k\}$. For this, suppose that the symbol $a'_i$ is in cycle $c'$. We pick arbitrarily another $r = \log m$ cycles $c'_1, \ldots, c'_r$ from the set $S$ (note that by our assumption, $\log m + 2 \leq k$). Now instead of using the set $S$, we use the set $S_0 = \{c', c'_1, \ldots, c'_r\}$. Since the set $S_0$ has $2^r - 1 \geq m - 1$ subsets $S'$ such that $S' \neq S_0$ and the cycle $c'$ is in $S'$, at least one of these subsets has not been used. These $2^r - 1$ subsets of $S_0$ can be enumerated in a systematic manner in time $O(r2^r) = O(m \log m) = O(n \log n)$ since each such subset contains at most $r = O(\log m)$ cycles. Now an unused subset of $S_0$ is also an unused subset of $S$, with which we can construct the paths $P_i$ and $P'_i$ in time $O(n)$. In conclusion, the paths $P_i$ and $P_i'$ can be constructed in time $O(n \log n)$ for each $i = 2, \ldots, m$. Since $m \leq n$, the total time needed to construct the paths $P_i$ and $P_i'$, for all $i = 2, \ldots, m$, is bounded by $O(n^2 \log n)$. $\square$

Combining Lemma 3.1 and Lemma 3.2, we obtain immediately the following theorem.

**Theorem 3.3** *Let $u = c_1 \cdots c_k e_1 \cdots e_m$, $c_1 = (a_1 a_2 \cdots a_d 1)$, be the node in the $n$-star network $S_n$ as described in Assumption 2.1. The bulk distance $Bdist(u)$ from the node $u$ to the identity node $\varepsilon$ is $dist(u) + 4$ if and only if $m > \min\{2^{k-2}, 1 + \sum_{i=2}^{k} |c_i|\}$.*

# 4 The maximum number of node-disjoint shortest paths

In this section we diverge to a slightly different problem. Let $u$ be the node in the $n$-star network $S_n$ as described in Assumption 2.1. How many node-disjoint shortest paths (of length $dist(u)$) can we find from $u$ to $\varepsilon$?

This problem is closely related to a combinatorial problem, called *Maximum Partition Matching*, formulated as follows:

> Let $\mathcal{S} = \{C_1, C_2, \ldots, C_k\}$ be a collection of subsets of the universal set $U = \{1, 2, \ldots, n\}$ such that $\bigcup_{i=1}^{k} C_i = U$, and $C_i \cap C_j = \phi$ for all $i \neq j$. A *partition matching* (of order $m$) of $\mathcal{S}$ consists of two ordered subsets $L = \{a_1, a_2, \ldots, a_m\}$ and $R = \{b_1, b_2, \ldots, b_m\}$ of $m$ elements of $U$ (the subsets $L$ and $R$ may not be disjoint), together with a sequence of $m$ distinct partitions of $\mathcal{S}$: $(\mathcal{A}_1, \mathcal{B}_1)$, $(\mathcal{A}_2, \mathcal{B}_2)$, ..., $(\mathcal{A}_m, \mathcal{B}_m)$ such that for all $i = 1, \ldots, m$, $a_i$ is contained in a subset in the collection $\mathcal{A}_i$ and $b_i$ is contained in a subset in the collection $\mathcal{B}_i$. The *Maximum Partition Matching* problem is to construct a partition matching of order $m$ for a given collection $\mathcal{S}$ with $m$ maximized.

**Theorem 4.1** *The Maximum Partition Matching problem is solvable in time $O(n^2 \log n)$.*

PROOF. An $O(n^2 \log n)$ time algorithm has been developed in [7] that, given a collection $\mathcal{S}$ of subsets in $U = \{1, 2, \ldots, n\}$, constructs a maximum partition matching of $\mathcal{S}$. We refer our readers to [7] (and [6]) for details. $\square$

Now we show how Theorem 4.1 can be used to find the maximum number of node-disjoint shortest paths from a node $u$ to the identity node $\varepsilon$ in star networks.

**Lemma 4.2** *Let $u = c_1 \cdots c_k e_1 \cdots e_m$, $c_1 = (a_1 a_2 \cdots a_d 1)$, be the node in the $n$-star network $S_n$ as described in Assumption 2.1. Then the number of node-disjoint shortest paths from $u$ to $\varepsilon$ cannot be larger than $1 + \sum_{i=2}^{k} |c_i|$.*

PROOF. According to *Rule 2* of the Shortest Path Rules, only the path leaving $u$ with a symbol in the set $\{a_2\} \cup \bigcup_{i=2}^{k} c_i$ can be a shortest path from $u$ to $\varepsilon$. $\square$

Another upper bound on the number of node-disjoint shortest paths from $u$ to $\varepsilon$ can be derived in terms of the maximum partition matching of the collection $\mathcal{S} = \{c_2, \ldots, c_k\}$ of nontrivial cycles in $u$, where each nontrivial cycle is regarded as a set of symbols.

**Lemma 4.3** *Let $u = c_1 \cdots c_k e_1 \cdots e_m$, $c_1 = (a_1 a_2 \cdots a_d 1)$, be the node in the $n$-star network $S_n$ as described in Assumption 2.1. Then the number of node-disjoint shortest paths from $u$ to $\varepsilon$ cannot be larger than 2 plus the number of partitions in a maximum partition matching in the collection $\mathcal{S} = \{c_2, \ldots, c_k\}$, where the cycles $c_i$ are regarded as sets of symbols.*

PROOF. Let $P_1, \ldots, P_s$ be $s$ node-disjoint shortest paths from $u$ to $\varepsilon$. For each path $P_i$, let $u_i$ be the first node on $P_i$ such that in $u_i$ the primary cycle is a trivial cycle. The node $u_i$ is obtained by repeatedly applying *Rule 2* of the Shortest Path Rules, starting from the node $u$. It is easy to prove, by induction, that in any node $v$ on the subpath from $u$ to $u_i$ on the path $P_i$, the only possible nontrivial cycle that is not in $\{c_2, \ldots, c_k\}$ is the primary cycle. In particular, the node $u_i$ must have a cycle structure of the form

$$u_i = c'_1 \cdots c'_{k_i} e'_1 \cdots e'_{m_i}$$

where $c'_i$ are nontrivial cycles, $e'_j$ are trivial cycles and $\mathcal{B}_i = \{c'_1, \ldots, c'_{k_i}\}$ is a subcollection of the collection $\mathcal{S} = \{c_2, \ldots, c_k\}$.

Assume that the path $P_i$ leaves $u$ with the symbol $b_i$. By *Rule 2* in the Shortest Path Rules, $b_i$ is either $a_2$ or one of the symbols in $\cup_{i=2}^{k} c_i$. Moreover, also by this rule, once $b_i$ is contained in the primary cycle in a node in the path $P_i$, it will stay in the primary cycle in the nodes along the path $P_i$ until it is deleted from the primary cycle, i.e., until $b_i$ is contained in a trivial cycle. In particular, the symbol $b_i$ is not in the set $\cup_{j=1}^{k_i} c'_j$.

Now suppose that the path $P_i$ enters the node $\varepsilon$ with a symbol $d_i$. Thus, the node $w_i$ before $\varepsilon$ on the path $P_i$ must have a cycle structure $(d_i 1)$ if we discard trivial cycles in $w_i$. Since the symbol $d_i$ is in a nontrivial cycle in $w_i$, by Fact 2.2, $d_i$ is also in a nontrivial cycle in the node $u_i$, that is $d_i \in \cup_{j=1}^{k_i} c'_j$. The only exception is $d_i = a_d$ (in this case $u_i = \varepsilon$).

Now we let $\mathcal{A}_i = \mathcal{S} - \mathcal{B}_i$. Then we can conclude that except for at most two paths $P_1$ and $P_2$, each $P_i$, $i \geq 3$, of the other paths $P_3, \ldots, P_s$ must leave the node $u$ with a symbol $b_i$ in $\mathcal{A}_i$ and enter $\varepsilon$ with a symbol $d_i$ in $\mathcal{B}_i$. (The two exceptional paths $P_1$ and $P_2$ may leave $u$ with the symbol $a_2$ or enter $\varepsilon$ with the symbol $a_d$.)

Now since the $s$ paths $P_1, \ldots, P_s$ are node-disjoint, the symbols $b_3, \ldots, b_s$ are all pairwise distinct, and the symbols $d_3, \ldots, d_s$ are also pairwise distinct. Moreover, since all nodes $u_3, \ldots, u_s$ are pairwise distinct, the collections $\mathcal{B}_3, \ldots, \mathcal{B}_s$ of cycles are also pairwise distinct. Consequently, the partitions $(\mathcal{A}_3, \mathcal{B}_3), \ldots, (\mathcal{A}_s, \mathcal{B}_s)$ of the collection $\mathcal{S} = \{c_2, \ldots, c_k\}$, together with the symbol sets $\{b_3, \ldots, b_s\}$ and $\{d_3, \ldots, d_s\}$, form a partition matching of the collection $\mathcal{S}$.

This concludes that $s$ cannot be larger than 2 plus the number of partitions in a maximum partition matching of the collection $\mathcal{S} = \{c_2, \ldots, c_k\}$, thus proves the lemma. □

Now we show how we construct a maximum number of node-disjoint shortest paths from the node $u = c_1 \cdots c_k e_1 \cdots e_m$, $c_1 = (a_1 \cdots a_d 1)$, as described in Assumption 2.1, to the identity node $\varepsilon$ in the $n$-star network $S_n$. We first show how to route a single shortest path from $u$ to $\varepsilon$, given a partition $(\mathcal{A}, \mathcal{B})$ of the collection $\mathcal{S} = \{c_2, \ldots, c_k\}$, and a pair of symbols $b$ and $d$, where $b$ is in a cycle in $\mathcal{A}$ and $d$ is in a cycle in $\mathcal{B}$. We also allow $b$ to be $a_2$ — in this case $d$ must be in $\cup_{i=2}^{k} c_i$ and $\mathcal{A} = \phi$, and $\mathcal{B} = \mathcal{S}$. Similarly, we allow $d$ to be $a_d$ — in this case $b$ must be in $\cup_{i=2}^{k} c_i$ and $\mathcal{B} = \phi$, and $\mathcal{A} = \mathcal{S}$. Consider the algorithm SINGLE ROUTING given in Figure 1. Since the algorithm SINGLE ROUTING starts with the node $u$ and applies only transpositions described in the Shortest Path Rules, we conclude that the algorithm SINGLE ROUTING constructs a shortest path from the node $u$ to the node $\varepsilon$.

Now we are ready for describing the final algorithm. Consider the algorithm MAXIMUM SHORTEST ROUTING given in Figure 2.

**Algorithm.** Single Routing

Input: A partition $(\mathcal{A}, \mathcal{B})$ of $\mathcal{S} = \{c_2, \ldots, c_k\}$, and two symbols $b$ and $d$, where $b$ is in a cycle in $\mathcal{A}$ and $d$ is in a cycle in $\mathcal{B}$. $b$ can be $a_2$ with $\mathcal{A} = \phi$, and $d$ can be $a_d$ with $\mathcal{B} = \phi$.

Output: A shortest path from $u$ to $\varepsilon$ leaving $u$ with $b$ and entering $\varepsilon$ with $d$.

1. **if** $b \neq a_2$

   apply $\pi[b]$ to $u$ to merge the cycle in $\mathcal{A}$ that contains $b$ into the primary cycle $c_1$; then merge in an arbitrary order the rest of the cycles in $\mathcal{A}$ into the primary cycle;
2. repeatedly delete symbols in the primary cycle until the primary cycle becomes a trivial cycle;
3. **if** $d \neq a_d$

   suppose that the cycle $c$ containing $d$ in $\mathcal{B}$ is $c = (d' \cdots d)$. Apply $\pi[d']$ to merge $c$ into the primary cycle (1); then merge in an arbitrary order the rest of the cycles in $\mathcal{B}$ into the primary cycle;
4. repeatedly delete symbols in the primary cycle until reach the node $\varepsilon$;

Figure 1: The algorithm Single Routing

**Algorithm.** Maximum Shortest Routing

Input: The node $u$ in the $n$-star network $S_n$, as described in Assumption 2.1.

Output: A maximum number of node-disjoint shortest paths from $u$ to $\varepsilon$.

1. Construct a maximum partition matching $M[(b_1, d_1), \ldots, (b_s, d_s)]$ in $\mathcal{S} = \{c_2, \ldots, c_k\}$ with the partitions $(\mathcal{A}_1, \mathcal{B}_1)$, …, $(\mathcal{A}_s, \mathcal{B}_s)$ of $\mathcal{S}$;
2. **if** $s = \sum_{i=2}^{k} |c_i|$ **then** construct $s + 1$ node-disjoint shortest paths as follows.
   - 2.1. Call the algorithm Single Routing with the partition $(\phi, \mathcal{S})$ of $\mathcal{S}$ and the symbol pair $(a_2, d_1)$;
   - 2.2. Call the algorithm Single Routing with the partition $(\mathcal{S}, \phi)$ of $\mathcal{S}$ and the symbol pair $(b_1, a_d)$;
   - 2.3. For $i = 2$ to $s$, call the algorithm Single Routing with the partition $(\mathcal{A}_i, \mathcal{B}_i)$ of $\mathcal{S}$ and the symbol pair $(b_i, d_i)$;
3. **if** $s < \sum_{i=2}^{k} |c_i|$ **then** construct $s + 2$ node-disjoint shortest paths as follows.
   - 3.1. Let $b_0, d_0 \in \cup_{i=2}^{k} c_i$ such that $b_0 \neq b_i$, $i = 1, 2, \ldots, s$ and $d_0 \neq d_i$, $i = 1, 2, \ldots, s$;
   - 3.2. Call the algorithm Single Routing with the partition $(\phi, \mathcal{S})$ of $\mathcal{S}$ and the symbol pair $(a_2, d_0)$;
   - 3.3. Call the algorithm Single Routing with the partition $(\mathcal{S}, \phi)$ of $\mathcal{S}$ and the symbol pair $(b_0, a_d)$;
   - 3.4. For $i = 1$ to $s$, call the algorithm Single Routing with the partition $(\mathcal{A}_i, \mathcal{B}_i)$ of $\mathcal{S}$ and the symbol pair $(b_i, d_i)$;

Figure 2: The algorithm Maximum Shortest Routing

**Theorem 4.4** *The algorithm* MAXIMUM SHORTEST ROUTING *constructs a maximum number of node-disjoint shortest paths from the node $u$ to the identity node $\varepsilon$ in time $O(n^2 \log n)$.*

PROOF. From Lemma 4.2 and Lemma 4.3, we know that the number of shortest paths constructed by the algorithm MAXIMUM SHORTEST ROUTING matches the maximum number of node-disjoint shortest paths from $u$ to $\varepsilon$. What remains is to show that all these paths are node-disjoint.

Suppose that the algorithm MAXIMUM SHORTEST ROUTING constructs $h$ shortest paths $P_1$, $P_2$, ..., $P_h$ from node $u$ to node $\varepsilon$, here $h = s+1$ or $h = s+2$ depending on whether $s = \sum_{i=2}^{k} |c_i|$, and suppose that the path $P_i$ is constructed by calling the algorithm SINGLE ROUTING on partition $(\mathcal{A}_i, \mathcal{B}_i)$ of $\mathcal{S}$ and symbol pair $(b_i, d_i)$ for all $i = 1, \ldots, h$. Note that if $\mathcal{A}_i = \phi$ then we have $b_i = a_2$, and if $\mathcal{B}_i = \phi$ then we have $d_i = a_d$. Now fix an $i$ and consider the path $P_i$, which is constructed from the partition $(\mathcal{A}_i, \mathcal{B}_i)$ and the symbol pair $(b_i, d_i)$. Let $\mathcal{A}_i = \{c_2^{(i)}, \ldots, c_l^{(i)}\}$ and $\mathcal{B}_i = \{c_{l+1}^{(i)}, \ldots, c_k^{(i)}\}$, where if $\mathcal{A}_i \neq \phi$ then the cycle $c_2^{(i)}$ is of form $c_2^{(i)} = (b_i \cdots b_i')$ and if $\mathcal{B}_i \neq \phi$ then the cycle $c_{l+1}^{(i)}$ is of form $c_{l+1}^{(i)} = (\cdots d_i)$. Finally, recall that the primary cycle $c_1$ has the form $c_1 = (a_1 a_2 \cdots a_d 1)$.

The interior nodes of the path $P_i$ can be split into three segments $I_1^{(i)}$, $I_2^{(i)}$, and $I_3^{(i)}$. The first segment $I_1^{(i)}$ corresponds to nodes constructed in step 1 of the algorithm SINGLE ROUTING that first merges cycle $c_2^{(i)}$ into the primary cycle $c_1$, obtaining a cycle of form $(b_i \cdots b_i' a_1 a_2 \cdots a_d 1)$, then merges cycles $c_3^{(i)}$, ..., $c_l^{(i)}$ into the primary cycle. Therefore, for all nodes in this segment, the primary cycle is of the form

$$(\cdots b_i \cdots b_i' a_1 a_2 \cdots a_d 1)$$

The second segment $I_2^{(i)}$ corresponds to the nodes constructed by step 2 of the algorithm SINGLE ROUTING that deletes symbols in the primary cycle. All nodes in this segment are of the form

$$(\cdots 1) c_{l+1}^{(i)} \cdots c_k^{(i)}$$

The third segment $I_3^{(i)}$ corresponds to the nodes constructed by step 3 and step 4 of the algorithm SINGLE ROUTING, which first merges the cycle $c_{l+1}^{(i)}$ into the primary cycle (1), obtaining a cycle of form $(\cdots d_i 1)$, then merges the cycles $c_{l+2}^{(i)}$, ..., $c_k^{(i)}$ into the primary cycle, and then deletes symbols in the primary cycle. Therefore, in all nodes in this segment, the primary cycle should have the form

$$(\cdots d_i 1)$$

In case $\mathcal{A}_i = \phi$, we have $b_i = a_2$ and the segment $I_1^{(i)}$ is empty, and in case $\mathcal{B}_i = \phi$, we have $d_i = a_d$ and the segment $I_3^{(i)}$ is empty.

We now show that any two shortest paths $P_i$ and $P_j$, $i \neq j$, constructed by the algorithm MAXIMUM SHORTEST ROUTING are node-disjoint. Let $v$ be a node on the path $P_i$.

Suppose that $v = (\cdots b_i \cdots b_i' a_1 a_2 \cdots a_d 1) \cdots$ is a node on the first segment $I_1^{(i)}$ of the path $P_i$. The node cannot be on the first segment $I_1^{(j)}$ of the path $P_j$ since all nodes on $I_1^{(j)}$ are of form $(\cdots b_j \cdots b_j' a_1 a_2 \cdots a_d 1) \cdots$ and $b_i \neq b_j$ (thus $b_i' \neq b_j'$). Moreover, the node $v$ cannot be on the second or the third segment of $P_j$ since the cycle structure of a node on the second or the third segment of $P_j$ has more trivial cycles (note that each execution of step 2 of the algorithm SINGLE ROUTING creates a new trivial cycle in the cycle structure).

If $v = (\cdots 1)c^{(i)}_{l+1} \cdots c^{(i)}_k$ is on the second segment $I^{(i)}_2$ of the path $P_i$, then $v$ cannot be on the second segment $I^{(j)}_2$ of $P_j$ since each node on $I^{(j)}_2$ is of form $(\cdots 1)c^{(j)}_{f+1} \cdots c^{(j)}_k$ and

$$\mathcal{B}_i = \{c^{(i)}_{l+1}, \ldots, c^{(i)}_k\} \neq \mathcal{B}_j = \{c^{(j)}_{f+1}, \ldots c^{(j)}_k\}$$

The node $v$ can neither be on the third segment $I^{(j)}_3$ of $P_j$ since each node on the segment $I^{(j)}_3$ is of form $(\cdots d_j 1)\cdots$, where $d_j \in \cup_{i=2}^k c_i$ while the primary cycle in the node $v$ is either a trivial cycle or of form $(\cdots a_d 1)$ where $a_d$ is in $c_1$.

Finally, if $v = (\cdots d_i 1)\cdots$ is on the third segment of the path $P_i$, then $v$ cannot be on the third segment of $P_j$ because $d_i \neq d_j$.

By symmetry, the above analysis shows that the two shortest paths $P_i$ and $P_j$ constructed by the algorithm Maximum Shortest Routing must be node-disjoint.

The running time of the algorithm Maximum Shortest Routing is dominated by step 1 of the algorithm, which takes time $O(n^2 \log n)$ according to Theorem 4.1. Thus, the algorithm Maximum Shortest Routing runs in time $O(n^2 \log n)$. □

Construction of the maximum number of node-disjoint shortest paths between two nodes in star networks was previously studied in [16], which presents an algorithm that runs in exponential time in the worst case. More seriously, the algorithm seems based on an incorrect observation, which claims that when there are more than one nontrivial cycles in a node $u$, the maximum number of node-disjoint shortest paths from $u$ to $\varepsilon$ is always an even number. Therefore, the algorithm in [16] always produces an even number of node-disjoint shortest paths from $u$ to $\varepsilon$. A counterexample to this observation has been constructed in [6].

## 5 Conclusion: an optimal parallel routing

Combining all the previous discussion in the present paper, we obtain an $O(n^2 \log n)$ time algorithm, Optimal Parallel Routing as shown in Figure 3, that constructs $n-1$ node-disjoint paths of bulk length $Bdist(u)$ from any node $u$ to the identity node $\varepsilon$ in the $n$-star network $S_n$.

The correctness of the algorithm Optimal Parallel Routing has been proved by Lemma 3.1, Lemma 3.2, Theorem 4.4, and the results in [10]. The running time of the algorithm is $O(n^2 \log n)$.

We would like to make a few remarks on the complexity of our algorithm. The bulk distance problem on general graphs is NP-hard [13]. Thus, it is very unlikely that the bulk distance problem can be solved in time polynomial in the size of the input graph. On the other hand, our algorithm solves the bulk distance problem in time $O(n^2 \log n)$ on the $n$-star network. Note that the $n$-star network has $n!$ nodes. Therefore, the running time of our algorithm is actually a polynomial of the logarithm of the size of the input star network. Moreover, our algorithm is almost optimal (differs at most by a $\log n$ factor) since the following lower bound can be easily observed — the distance $dist(u)$ from $u$ to $\varepsilon$ can be as large as $\Theta(n)$. Thus, constructing $n-1$ node-disjoint paths from $u$ to $\varepsilon$ takes time at least $\Theta(n^2)$ in the worst case.

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**Algorithm.** OPTIMAL PARALLEL ROUTING

INPUT: A node $u$ in the $n$-star network $S_n$.
OUTPUT: $n-1$ node-disjoint paths of length $Bdist(u)$ from $u$ to $\varepsilon$.

1. **if** the primary cycle in $u$ is a trivial cycle
   **then** apply the algorithm described in Section 2 to construct $n-1$ node-disjoint paths of bulk length $Bdist(u)$; STOP
2. { At this point, we assume that $u$ satisfies the conditions in Assumption 2.1.}
   **case 1** $m > \min\{2^{k-2}, 1 + \sum_{i=2}^{k} |c_i|\}$
   use Day and Tripathi's algorithm [10] to construct $n-1$ node-disjoint paths of bulk length $dist(u) + 4 = Bdist(u)$;
   **case 2** $m \leq \min\{2^{k-2}, 1 + \sum_{i=2}^{k} |c_i|\}$
   apply the algorithm MAXIMUM SHORTEST ROUTING to construct the maximum number of node-disjoint shortest paths from $u$ to $\varepsilon$;
   **if** the algorithm MAXIMUM SHORTEST ROUTING returns $n-1$ paths
   **then** these are $n-1$ node-disjoint paths of bulk length $dist(u) = Bdist(u)$
   **else** apply the algorithm in Lemma 3.2 to construct $n-1$ node-disjoint paths of bulk length $dist(u) + 2 = Bdist(u)$

Figure 3: The algorithm OPTIMAL PARALLEL ROUTING


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