Linearly Synthesizing
2-Connected Simplicial Graphs

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Abstract

It is proved that for any two 2-connected, smooth, and simplicial graphs G and H such that H is homeomorphic to a subgraph of G, there is a sequence of 2-connected subgraphs $G_0 \subseteq G_1 \subseteq \cdots \subseteq G_r = G$ of G such that G_0 is homeomorphic to the graph H, each G_i is obtained from G_{i-1} by adding a chain, $1 \leq i \leq r$, and for any two consecutive graphs in the sequence, at least one is homeomorphic to a smooth and simplicial graph. Our result has found applications in the study of graph imbeddings on topological surfaces and in designing efficient graph algorithms.

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1 Introduction

Let G be a graph[3]. A *suspended chain* (or simply, a *chain*) C of G is a simple open path in G such that every interior vertex of C has valence 2 in G. We observe that a single edge that is not a self-loop is a suspended chain without interior vertices.

By *removing* a suspended chain C from a graph G, we mean removing all of its edges and interior vertices, but not its two end-vertices. In his classical paper [20], Whitney proved that for any loopless 2-connected graph except a simple cycle, there is a suspended chain whose removal yields a 2-connected graph.

Thus, given a loopless 2-connected graph, we can repeatedly delete proper suspended chains until the remaining graph is a simple cycle. Conversely, we can synthesize any loopless 2-connected graph from a simple cycle by successively adding suspended chains, so that at every stage of the construction we have a 2-connected graph.

Whitney's theorem has the following nature: Given a graph G with a property $\mathcal{P}$ and a subgraph H of G which also has the property $\mathcal{P}$, the graph G can be obtained from H by performing some simple operations, such as chain addition, such that at every stage of the construction, the property $\mathcal{P}$ is preserved. One may readily perceive the importance of Whitney-type theorem to inductive proofs for graph problems. Recently, Whitney's theorem has also been profitably used in designing parallel algorithms [11, 13].

Whitney-type theorems have also been studied for higher connectivity graphs. Tutte [17] showed that every 3-connected graph G can be obtained from a wheel by chain additions and vertex splittings, while preserving the 3-connectedness during the construction. Negami [14] pointed out that Tutte's construction can be started from any 3-connected minor of G. Slater [15] proved that every 4-connected graph can be obtained from the complete graph K_5 by chain additions, 4-solderings, 4-point-splittings, 4-line-splittings, and 3-fold-4-point-splittings, while preserving the 4-connectedness. Kelmans [9] showed that every "topologically 3-connected" graph G can be obtained from any "topologically 3-connected" subgraph by chain additions while "topological 3-connectedness" is preserved. More generalized versions of Whitney-type theorem for 3-connected matroids have also been derived [16, 1, 2].

[3]Throughout this paper, a graph may have multiple adjacencies or self-adjacencies. It is taken to be connected, unless one can readily infer otherwise from the immediate context.

In this paper, we shall derive some new Whitney-type theorems for 2-connected simplicial graphs. We would like to preserve 2-connectedness and simplicialness *simultaneously* when we synthesize a 2-connected simplicial graph from a 2-connected simplicial subgraph. This does not seem to make much sense at first glance. After all, every subgraph of a simplicial graph is simplicial. However, simplicialness is no longer inherited if we concentrate on the topological properties of a graph, where all 2-valent vertices are smoothed out.

Intuitively, we regard a graph G as "intrinsically simplicial" if the graph obtained from G by smoothing all 2-valent vertices is simplicial. Our main theorem in this paper is that for any two 2-connected and "intrinsically simplicial" graphs G and H such that H is homeomorphic to a subgraph of G, there is a sequence of 2-connected subgraphs $G_0, G_1, \cdots, G_r$ of G such that G_0 is homeomorphic to the graph H, $G_r = G$, that each G_i is obtained from G_{i-1} by adding a suspended chain, $1 \le i \le r$, and that for every two consecutive graphs in the sequence, at least one of them is "intrinsically simplicial".

Our results are particularly useful to the study of graph imbeddings on topological surfaces, where a graph is regarded as the carrier of a 1-complex, where all the 2-valent vertices can be ignored, and where the intrinsic simplicialness plays an important role. The reader is referred to the papers [3, 6, 4, 5] for the applications. Recently, our techniques and results have also found applications in designing efficient graph algorithms [7, 8].

2 Linear synthesis of graphs

Linear synthesis of graphs[4] originates in concept in the classical work of Whitney [20].

Definition A *linear synthesis* of a graph G from a subgraph G_0 is a sequence of increasing subgraphs $G_0, G_1, \cdots, G_{r-1}, G_r = G$ such that for $i = 1, \cdots, r$, the graph complement $G_i - G_{i-1}$ is a suspended chain in G_i, with its two end-vertices in the graph G_{i-1}.

Let u be a 2-valent vertex of a graph G with neighbors v and w, different from u but not necessarily different from each other. By *smoothing* the vertex u in G, we mean replacing the vertex u and its incident edges by a single

[4] In some literature, the inverse of linear synthesis is called *ear decomposition* of a graph.

edge between v and w. One cannot smooth the only vertex of a 1-cycle, even through it is 2-valent, because it has no neighbors except itself.

It is a routine exercise to demonstrate that two graphs are *homeomorphic* if smoothing all 2-valent vertices in both graphs yields isomorphic graphs. Suppose that a graph G has a linear synthesis from a subgraph H and that H_0 is a graph homeomorphic to H. For simplicity, we also say that the graph G has a linear synthesis from the graph H_0.

A graph is called *simplicial* if it contains neither multiple edges nor self-loops. A graph is called *smooth* if it does not contain 2-valent vertices. In particular, a 1-cycle is neither simplicial nor smooth.

Definition Let G be a graph that is not a cycle. The *smooth model* of G is the smooth graph that is homeomorphic to G. The smooth model of a cycle is the 1-cycle. A graph is *smoothly k-connected simplicial* if its smooth model is k-connected and simplicial.

Therefore, the smooth model of a graph G can be obtained from G by smoothing all 2-valent vertices. The objective of introducing the smooth model of a graph is to focus on the topological properties of a graph. From this point of view, all 2-valent vertices in a graph are ignored. By our definition, a graph that is not a cycle is 2-connected if and only if it has a 2-connected smooth model, and, a graph G has a simplicial smooth model if and only if every graph homeomorphic to G is simplicial.

Definition A graph G has a *k-connected linear synthesis from a graph H* (or simply, a *$(k;H)$-linear synthesis*) if G has a linear synthesis $G_0, G_1, \cdots, G_r = G$ such that G_0 is homeomorphic to H and each graph in the sequence has a k-connected smooth model.

Definition A graph G has a *k-connected and semi-simplicial linear synthesis with variation distance d from a graph H* (or simply, a *$(k,d;H)$-linear synthesis*) if G has a $(k;H)$-linear synthesis $G_0, G_1, \cdots, G_r = G$ such that for any d consecutive graphs in the sequence, at least one of them has a simplicial smooth model.

Note that the above definitions depend only on the homeomorphism type of a graph.

Whitney's theorem and Kelmans's theorem can be presented as follows:

Theorem 2.1 (Whitney) *Let C_0 be a simple cycle. A graph without self-loops is 2-connected if and only if it has a $(2; C_0)$-linear synthesis.*

Theorem 2.2 (Kelmans) *A graph G is 3-connected if and only if for every 3-connected graph H that is homeomorphic to a subgraph of G, G has a $(3; H)$-linear synthesis.*

The theory of bridges and attachments, as presented by Tutte [18] in his classic study of graph connectivity, is needed in our discussion. We slightly modify the terminology he used in [19].

Let H be a subgraph of a connected graph G. Denote by $G - H$ the graph obtained from G by first deleting all edges in H and then deleting all the isolated vertices. An *attachment* of H in G is a vertex that lies both in H and in $G - H$.

If some edge of $G - H$ has both its ends in the subgraph H, then it is called a *trivial bridge* of $G - H$. Let W be the (possibly not connected) subgraph of G obtained by deleting all vertices (and the edges incident on them) in $V(H)$ from G, and let D be a connected component of W. Let B be the subgraph of G obtained from D by adjoining to it each edge of G with one end in D and one in H, together with the end-vertex of that edge in H. Then B is called a *nontrivial bridge* of $G - H$. A subgraph B of G is called a *bridge* of $G - H$ if B is either a trivial bridge or a nontrivial bridge.

Let B be a bridge of $G - H$. Then B is a connected subgraph of G. Moreover, a nontrivial bridge with some of its attachments deleted is also a connected subgraph of G. If a vertex v belongs to two distinct bridges of $G - H$, then v must be an attachment of the subgraph H in G. These follow directly from the definitions. For further discussion of bridges and attachments, see Tutte [19]. We will also investigate some special properties of bridges and attachments of 2-connected graphs in the following sections.

3 Some lemmas for 2-connected graphs

We first prove a few lemmas for 2-connected graphs.

Lemma 3.1 *Let G be a 2-connected graph without self-loops, let e be an edge, and let v_1 and v_2 be two distinct vertices of G. Then there is a simple path p in G from v_1 to v_2 such that p contains the edge e.*

PROOF. Subdivide the edge e by a new vertex w_1, introduce another new vertex w_2 adjoined to the vertices v_1 and v_2, and call the resulting graph G'. It is easy to see from the definition of 2-connectedness that the graph G' is 2-connected. By Menger's theorem, there are two vertex-disjoint paths p_1 and p_2 adjoining the two vertices w_1 and w_2. Thus, the paths p_1 and p_2 form a simple cycle C in the graph G'. Now deleting the vertex w_2 and smoothing the vertex w_1 in C gives us the desired path in the graph G. □

Lemma 3.2 *Let G be a smoothly 2-connected simplicial graph, and let e be an edge of G. Then the edge e is contained in a subgraph of G that is homeomorphic to the complete graph K_4.*

Proof. Without loss of generality, we can suppose that the graph G itself is 2-connected, smooth, and simplicial (i.e., G is its own smooth model). Otherwise, we can first smooth all 2-valent vertices in G, and replace the edge e of G by the edge in the new graph to which the edge e belongs.

It is known that a simplicial graph with all its vertices of valence greater than 2 contains a homeomorphic copy of the complete graph K_4 (see [10], page 65). Now, the graph G does not contain 2-valent vertices, since G is smooth and is not a cycle, and G does not contain 1-valent vertices either, since G is 2-connected. Consequently, the graph G contains a subgraph H which is homeomorphic to the complete graph K_4.

If the subgraph H contains the edge e, then we are done. Otherwise, the edge e is contained in some bridge B of $G - H$. Since G is 2-connected and simplicial, the bridge B has at least two distinct attachments a and a'. By Lemma 3.1, the graph G contains a simple path p from a to a' containing the edge e. The path p may contain more than two attachments of B and may not be entirely contained in the bridge B. However, it is straightforward to see that we can always find a subpath p' on the path p such that (1) the path p' contains the edge e, (2) the path p' is entirely contained in the bridge B, (3) no interior vertices of p' are attachments of B, and (4) the two end-vertices of p' are two distinct attachments a_1 and a_2 of the bridge B. Now consider all possible cases of locating the two points a_1 and a_2 on the complete graph K_4. There are exactly six different such cases, up to homeomorphism, as illustrated in Figure 1. It is routine to check for each of the six cases, that adding the path p' to the K_4 and properly deleting an edge or a part of an edge of the K_4 will yield a homeomorphic copy of the complete graph K_4. In Figure 1 the segment marked with "×" in each graph is the part of K_4 to be deleted. □

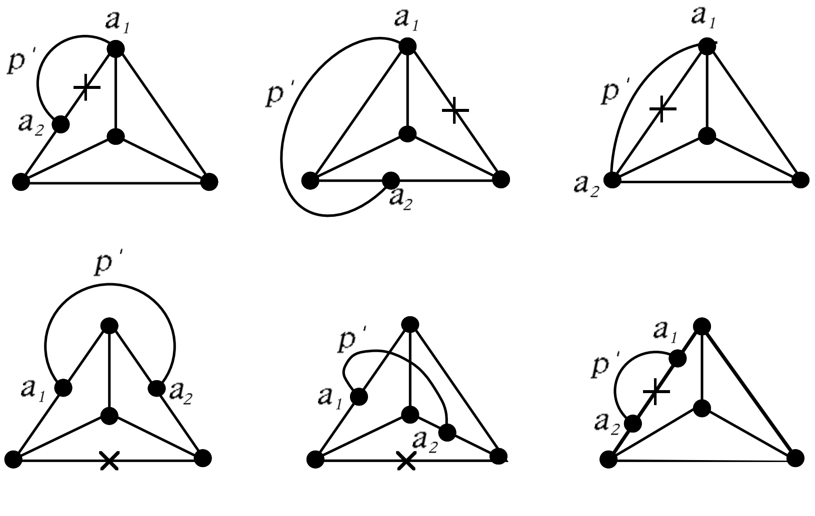

Figure 1: Replacing a segment of K_4 by p'

Let G be a graph, and let u and v be two vertices of G. We denote by $G_{+(u,v)}$ the graph obtained from G by adding an edge joining the two vertices u and v. The added edge is called *the new edge* of $G_{+(u,v)}$.

Lemma 3.3 *Let G be a 2-connected graph, and let H be a subgraph of G. Let B be a bridge of $G - H$ such that B has exactly two attachments a and a'. Then the graph $B_{+(a,a')}$ is 2-connected.*

PROOF. Denote the new edge in $B_{+(a,a')}$ by e. We prove the lemma by applying Menger's theorem. Let v_1 and v_2 be two arbitrary vertices in $B_{+(a,a')}$. If v_1 and v_2 are just the two attachments a and a' of B, then the new edge e and a simple path in B adjoining a and a' will form a pair of vertex-disjoint paths adjoining v_1 and v_2 in $B_{+(a,a')}$.

Therefore, we suppose that at least one of the vertices v_1 and v_2 is not an attachment of B. Since G is 2-connected, by Menger's theorem, there are two vertex-disjoint simple paths p_1 and p_2 in G that adjoin v_1 and v_2. If both paths p_1 and p_2 are entirely in B, then of course they are also entirely in $B_{+(a,a')}$. If, say, p_1 is not entirely in B, then p_1 must get out of B via one of the attachments of B, then come back to B via the other attachment of B. Since at least one of the vertices v_1 and v_2 is not an attachment of B, at least one of the attachments a and a' is an interior vertex of the path p_1. This forces the path p_2 to be entirely in B. Now if we replace the path p_1 in

G by the path in $B_{+(a,a')}$ that consists of the parts of p_1 that are in B plus the new edge e, then together with p_2, we obtain two vertex-disjoint paths in $B_{+(a,a')}$ that adjoin the two vertices v_1 and v_2.

Therefore, for any pair of vertices in $B_{+(a,a')}$, we can always find two vertex-disjoint paths in $B_{+(a,a')}$ adjoining them. By Menger's theorem, the graph $B_{+(a,a')}$ is 2-connected. $\square$

A *reduced contraction* of a simplicial graph G along an edge e is the result of deleting the edge e from G, identifying its endpoints, and then deleting all but one edge in each set of multiple edges that are incident on the new vertex. We denote by G/e the graph so obtained. The point of the reduction is to ensure that G/e is simplicial.

Lemma 3.4 *Let G be a 2-connected, smooth and simplicial graph, and let H be a 2-connected subgraph. Let B be a nontrivial bridge of $G - H$, such that B has exactly two attachments a and a'. Then the graph $B_{+(a,a')}$ contains a subgraph that is homeomorphic to the complete graph K_4 and contains the new edge $e = \{a, a'\}$ in $B_{+(a,a')}$.*

PROOF. We prove the lemma by induction on the number of vertices of the graph G.

A 2-connected, smooth, and simplicial graph has at least four vertices. Thus, the smallest such graph is the complete graph K_4, for which the lemma is trivially true.

Now suppose that G contains more than four vertices. By Lemma 3.3, the graph $B_{+(a,a')}$ is 2-connected. Now there are two possible cases.

If in the graph B, both vertices a and a' have valence greater than 1, then the graph $B_{+(a,a')}$ is actually simplicial and smooth. Moreover, $B_{+(a,a')}$ is not a simple cycle since vertices a and a' both have valence ≥ 3. By Lemma 3.2, $B_{+(a,a')}$ contains a subgraph that is homeomorphic to K_4 and contains the new edge e in $B_{+(a,a')}$.

On the other hand, suppose that at least one of the vertices a and a' has valence 1 in the bridge B. Without loss of generality, we assume that a is a 1-valent vertex in B. Let the edge of B that is incident on a be $e' = \{a, b\}$.

Consider the graph G/e'. Tutte ([19], page 68) has proved that if G is 2-connected and simplicial, then either the graph G/e' or the graph $G - \{e'\}$ is 2-connected. Now the graph $G - \{e'\}$ is surely not 2-connected because the attachment a' is a cutpoint. So the graph G/e' must be 2-connected. Moreover, the graph G/e' is also simplicial and smooth, since the graph G

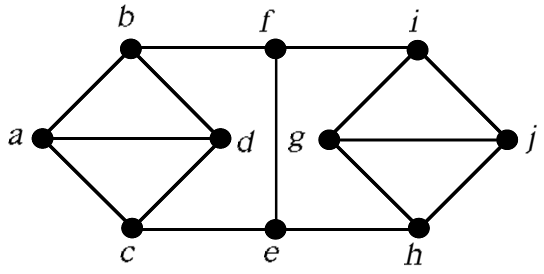

Figure 2: A 2-connected simplicial graph

is simplicial and smooth. The graph H is also a subgraph of G/e'. Finally, let B' be the graph B/e' if the vertices a' and b are not adjacent in G, or be the graph $(B - \{a', b\})/e'$ if $\{a', b\}$ is an edge of G. It is routine to check that the graph B' is a bridge of $G/e' - H$, such that B' has exactly two attachments a' and a'', where a'' is the new vertex in G/e' by identifying the two vertices a and b in G. B' is a nontrivial bridge of $G/e' - H$, since G contains neither 2-valent vertices nor multiple edges.

Thus, applying the inductive hypothesis on the graphs G/e', H, and B', we conclude that the graph ${B'}_{+(a',a'')}$ contains a homeomorphic copy of the complete graph K_4 that contains the new edge $e' = \{a', a''\}$ in ${B'}_{+(a',a'')}$. Now splitting the vertex a'' back into the two vertices a and b in $B_{+(a,a')}$ yields a homeomorphic copy of the complete graph K_4 in $B_{+(a,a')}$ that contains the new edge $e = \{a, a'\}$. □

4 Main theorem

Let G be a 2-connected simplicial graph. Then every subgraph of G is simplicial. Therefore, for any 2-connected linear synthesis of G, each graph in the sequence is simplicial. However, some graph in the sequence might *not* have a simplicial smooth model. An immediate question, therefore, is whether we can always build up a linear synthesis for a smoothly 2-connected simplicial graph from a smoothly 2-connected simplicial subgraph, such that all the graphs in the sequence are smoothly 2-connected simplicial.

The answer is "not quite but almost". For example, let G be the graph in Figure 2, which is 2-connected, smooth, and simplicial. Moreover, the subgraph of G induced by the vertices a, b, c, d, e, and f is homeomorphic to the complete graph K_4. It is easy to verify that G has no linear synthesis

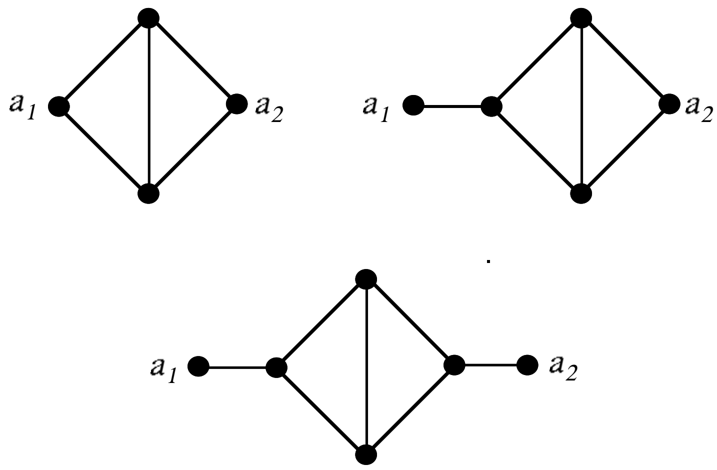

Figure 3: θ-rungs

from K_4 such that every graph in the sequence is smoothly 2-connected simplicial. On the other hand, if we are allowed to add more than one suspended chain at the same time, we will show that the 2-connectedness and the simplicialness of the smooth model can be preserved.

A graph is called a *θ-rung* if it is homeomorphic to one of the graphs in Figure 3. The two labeled vertices a_1 and a_2 in each θ-rung are called the *ends* of the θ-rung. Note that deleting an edge or a segment of an edge of the complete graph K_4 yields a θ-rung.

Lemma 4.1 *Let G and H be two smoothly 2-connected simplicial graphs such that G has a $(2,2;H)$-linear synthesis. Let G_θ be a graph obtained from G by attaching the two ends of a θ-rung to two points of an edge of G. Then G_θ is also smoothly 2-connected simplicial and has a $(2,2;H)$-linear synthesis.*

PROOF. Since the graph G is smoothly 2-connected simplicial, it is easy to see that the graph G_θ is also smoothly 2-connected simplicial.

Suppose that the two ends of a θ-rung are attached to two points b_1 and b_2 of an edge e of G. Suppose that a $(2,2;H)$-linear synthesis of G is

$$G_0, \cdots, G_{h-1}, G_h, \cdots, G_r = G$$

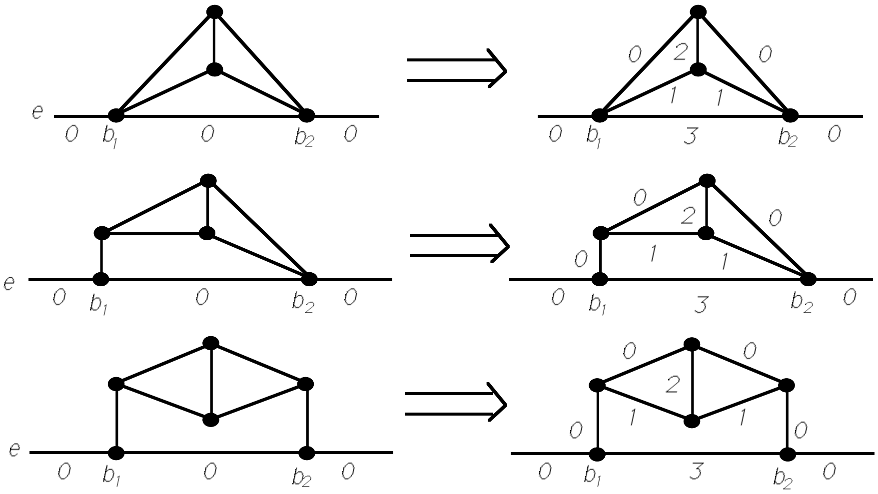

Figure 4: Attaching a θ-rung to the edge e

where G_0 is homeomorphic to H and that the graph G_h is the first graph in the sequence that contains the edge e. Consider the three different cases where three different θ-rungs are attached to the edge e, as illustrated by the left-side pictures in Figure 4, where the edge e is labeled by "0"s. In each of the cases, we replace the segment (b_1, b_2) on the edge e by a path p in the θ-rung adjoining the two ends. See the right-side pictures in Figure 4 where the path p in a θ-rung is labeled by "0"s.

Let G' be the graph obtained from G by replacing the segment (b_1, b_2) on e by the path p. The graph G' is obviously homeomorphic to the graph G. Moreover, if we let G'_j be the graph obtained from G_j by replacing the segment (b_1, b_2) on e by the path p, for $j = h, h+1, \cdots, r$, then the graphs G_j and G'_j are homeomorphic. The sequence

$$G_0, G_1, \cdots, G_{h-1}, G'_h, G'_{h+1}, \cdots, G'_r = G'$$

is obviously a $(2,2;H)$-linear synthesis of G'. (Note that if $h = 0$, then G_0 and G'_0 are different subgraphs of G_θ, but both of them are homeomorphic to H.)

Now we construct G_θ from G'. Let p_h be the path consisting of the edges labeled "h" in the right-side pictures in Figure 4, for $h = 1, 2, 3$, and inductively, let G'_{r+h} be the graph obtained by attaching the path p_h to the graph G'_{r+h-1}, for $h = 1, 2, 3$. It is straightforward to check that the graph G'_{r+2} is always smoothly 2-connected simplicial. Moreover, the graphs G'_r and G'_{r+3} are smoothly 2-connected simplicial, since they are homeomorphic to the smoothly 2-connected simplicial graphs G and G_θ, respectively. Therefore, the sequence

$$G_0, G_1, \cdots, G_{h-1}, G'_h, \cdots, G'_r, G'_{r+1}, G'_{r+2}, G'_{r+3} = G_\theta$$

is a $(2, 2; H)$-linear synthesis of G_θ. □

Now we are ready for the main theorem.

Theorem 4.2 *Let G and H be two smoothly 2-connected simplicial graphs such that H is homeomorphic to a subgraph of G. Then G has a $(2, 2; H)$-linear synthesis.*

PROOF. We assume without loss of generality that G and H are 2-connected, smooth, and simplicial graphs (otherwise, we work on the smooth models of G and H instead).

Let G_0 be a subgraph of G which is homeomorphic to the graph H. (Thus H is the smooth model of G_0.) Fix a homeomorphic mapping ϕ from G_0 to H. We say that a vertex v of G *belongs* to an edge e of H if v is on a suspended chain C of G_0 such that $\phi(C) = e$. Look at the bridges of $G - G_0$. We break our analysis into four different cases.

CASE 1. Two of the attachments a_1 and a_2 of some bridge B of $G - G_0$ do not belong to the same edge of the smooth model H.

Then the bridge B contains a simple path p from a_1 to a_2, such that no interior vertex of p is an attachment of G_0. It follows that the graph $G_1 = G_0 \cup \{p\}$ is smoothly 2-connected simplicial.

CASE 2. Some bridge B of $G - G_0$ has at least three attachments a_1, a_2, and a_3, and all of them belong to the same edge of the smooth model H.

Then B is a nontrivial bridge. We can find a simple path p_1 in B, such that the two end-vertices of p_1 are the two attachments a_1 and a_3, respectively, and no interior vertex of p_1 is an attachment of G_0. The path p_1 is not a single edge of G because p_1 belongs to the nontrivial bridge B of $G - G_0$. Since $B - \{a_1, a_3\}$ is a connected subgraph of B and G is smooth, we can find another simple path p_2 in B such that one end-vertex of p_2 is an interior vertex of p_1, that the other end-vertex of p_2 is the attachment a_2, and that no interior vertices of p_2 are attachments of G_0, or on the path p_1. Now the graph $G_1 = G_0 \cup \{p_1, p_2\}$ is smoothly 2-connected simplicial.

Moreover, the sequence $G_0, G_0 \cup \{p_1\}, G_1$ is a $(2,2;H)$-linear synthesis of G_1.

CASE 3. There exist two bridges B_1 and B_2, such that B_1 has two attachments a_1 and a_2, that B_2 has two attachments b_1 and b_2, that these four attachments are distinct and belong to a single edge e of the smooth model H, and that the attachments of different bridges occur on e in the order a_1, b_1, a_2, b_2.

In this case, we find a simple path p_1 in B_1 such that the two end-vertices of p_1 are the two attachments a_1 and a_2, respectively, and that no interior vertex of p_1 is an attachment of G_0; and we also find a simple path p_2 in B_2 such that the two end-vertices of p_2 are the two attachments b_1 and b_2, respectively, and that no interior vertex of p_2 is an attachment of G_0. The graph $G_1 = G_0 \cup \{p_1, p_2\}$ then is smoothly 2-connected simplicial. Moreover, the sequence $G_0, G_0 \cup \{p_1\}, G_1$ is a $(2,2;H)$-linear synthesis of G_1.

In any of the above three cases, we can find a smoothly 2-connected simplicial subgraph G_1 of G, which is a proper supergraph of G_0, such that G_1 has a $(2,2;H)$-linear synthesis.

If none of the above cases holds, then we must be able to find a bridge of $G - G_0$ that satisfies the following condition.

CASE 4. There is a bridge B that has exactly two attachments a_1 and a_2, and these two attachments are adjacent vertices in G_0.

First of all, in this case B cannot be a trivial bridge, lest B and the edge $\{a_1, a_2\}$ in G_0 form double adjacencies in G. Now by Lemma 3.4, the graph $B_{+(a_1,a_2)}$ contains a subgraph K that is homeomorphic to the complete graph K_4 and contains the new edge $e = \{a_1, a_2\}$ in $B_{+(a_1,a_2)}$, from which it follows that the graph $K' = K - \{e\}$ is homeomorphic to a θ-rung in B. Moreover, the smooth model of the graph $G_1 = G_0 + K'$ is obtained by attaching the two ends of the θ-rung to two points of an edge of the smooth model H. By Lemma 4.1, the graph G_1 is smoothly 2-connected simplicial and has a $(2,2;H)$-linear synthesis (note that the graph G_0 has a trivial $(2,2;H)$-linear synthesis).

Now the theorem can be proved by reiterating this analysis inductively on the graphs G and the smooth model of the graph G_1. $\square$

Corollary 4.3 *Let G be a smoothly 2-connected simplicial graph. Then G has a $(2,2;K_4)$-linear synthesis, where K_4 is the complete graph of four vertices.*

PROOF. By Lemma 3.2, the graph G contains a subgraph homeomorphic to the complete graph K_4. Now the corollary follows from Theorem 4.2. $\square$

Corollary 4.4 *Let G be a 2-connected, smooth, simplicial graph such that G has m edges. Then for any integer q, where $6 \le q \le m$, there is a 2-connected, smooth, simplicial graph H, such that H is homeomorphic to a subgraph of G, and that H has r edges, where $q \le r \le q+5$.*

PROOF. Let Γ be a $(2,2;K_4)$-linear synthesis of the graph G. Smoothing all 2-valent vertices of each graph in the sequence Γ, we get a sequence Γ' of graphs, each of that is 2-connected, smooth, and homeomorphic to a subgraph of the graph G. Moreover, the difference of edge numbers of two consecutive graphs G_{i-1} and G_i in the sequence Γ' is at most 3 since G_i is obtained from G_{i-1} by attaching an edge. Therefore, for any integer q, $6 \le q \le m$, at least two consecutive graphs in the sequence Γ' have their edge number between q and $q+5$, and at least one of them is simplicial. $\square$

Let H be a smoothly 2-connected simplicial subgraph of a smoothly 2-connected simplicial graph G. By Theorem 4.2, G has a $(2,2;H)$-linear synthesis $D_0, D_1, \cdots, D_r = G$, where D_0 is a subgraph homeomorphic to H. However, D_0 and H may be two different subgraphs of G. In fact, if CASE 4 is the case when we derive a $(2,2;H)$-linear synthesis for the graph $G_1 = G_0 + K'$ in the proof of Theorem 4.2, then we construct a graph G_0' from G_0 by replacing the segment (b_1, b_2) on the edge e in G_0 by the path p in K' (see the proof of Lemma 4.1). Then we construct a $(2,2;H)$-linear synthesis of G_1 whose first graph is G_0'. A question is: can we construct a $(2,2;H)$-linear synthesis of G whose first graph is *actually* H?

The construction is unfortunately not always possible. Again let the graph in Figure 2 be G and let H be the subgraph of G induced by the vertices a, b, c, d, e, and f. It is easy to verify that the graph G has no $(2,2;H)$-linear synthesis whose first graph is H.

To obtain a 2-connected and semi-simplicial linear synthesis of a smoothly 2-connected simplicial graph such that the first subgraph in the sequence is a specific subgraph, we need to further relax the restriction on variation distance in linear synthesis.

Lemma 4.5 *Let G be a smoothly 2-connected simplicial graph and let G_θ be a graph obtained from G by attaching the two ends of a θ-rung to two points of an edge of G. Then G_θ is also smoothly 2-connected simplicial and has a $(2,3;G)$-linear synthesis whose first graph is G.*

PROOF. Look at the pictures in Figure 4. It is easy to see that each of the θ-rungs can be decomposed into three paths p_1, p_2, and p_3 such that

$$G, G \cup \{p_1\}, G \cup \{p_1, p_2\}, G \cup \{p_1, p_2, p_3\} = G_\theta$$

is a $(2, 3; G)$-linear synthesis of G_θ whose first graph is G. $\square$

Theorem 4.6 *Let G be a smoothly 2-connected simplicial graph, and let H by a smoothly 2-connected simplicial subgraph of G. Then G has a $(2, 3; H)$-linear synthesis whose first graph is H.*

PROOF. The proof goes exactly the same as that of Theorem 4.2 except that in CASE 4, instead of constructing a $(2, 2; H)$-linear synthesis of G_1 whose first graph is homeomorphic to H and G_0, we construct a $(2, 3; H)$-linear synthesis of G_1 (by Lemma 4.5) whose first graph is actually G_0. Moreover, note that in the first three cases, we always construct a $(2, 3; H)$-linear synthesis of G_1 whose first graph is G_0. In the inductive step, note that a $(2, 3; G_0)$-linear synthesis of G_1 whose first graph is G_0 plus a $(2, 3; G_1)$-linear synthesis of G whose first graph is G_1 gives a $(2, 3; G_0)$-linear synthesis of G whose first graph is G_0. $\square$

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